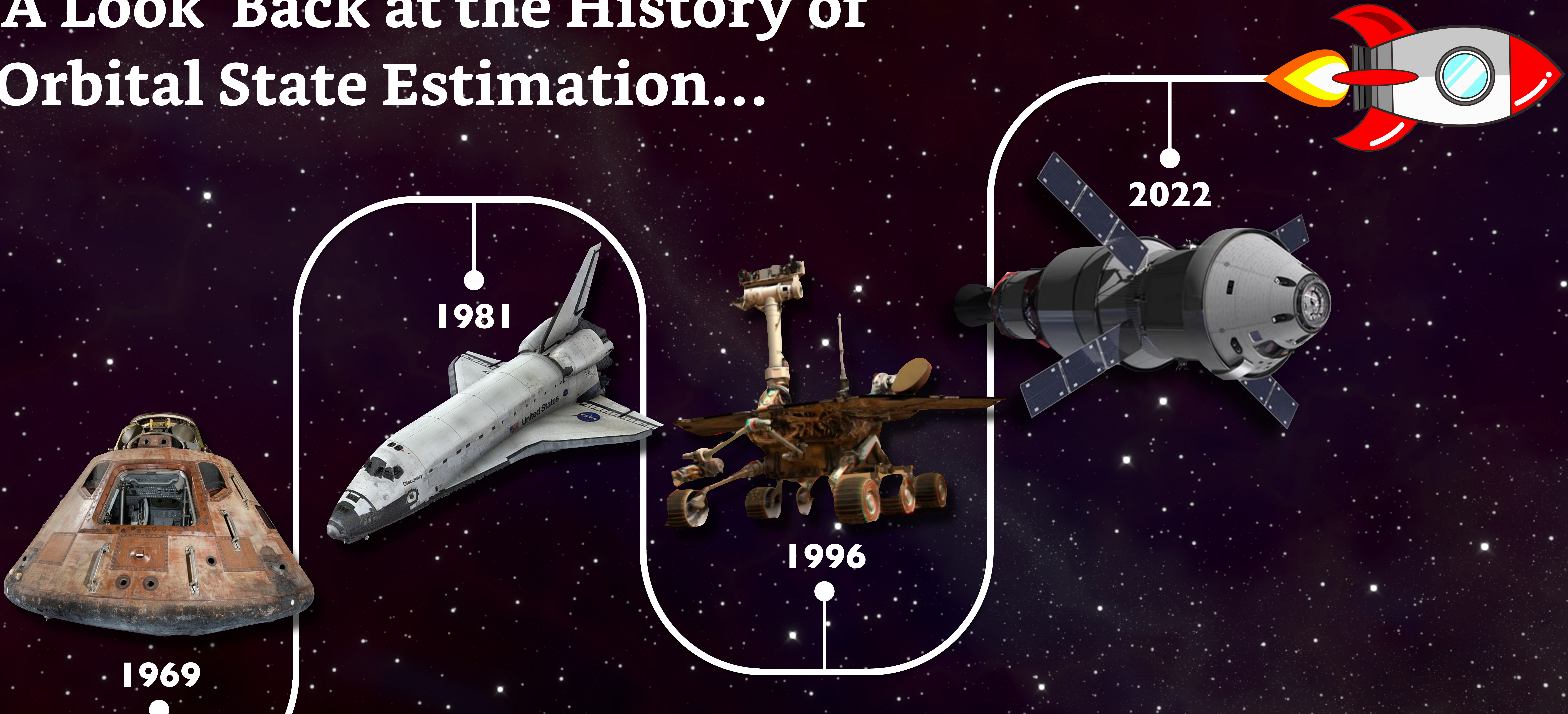
State Estimation of Chaotic Trajectories: A Higher-Dimensional, Grid-Based, Bayesian Approach to Uncertainty Propagation

Benjamin L. Hanson

Ph.D. student, Jacobs School of Engineering
Department of Mechanical and Aerospace Engineering
UC San Diego, La Jolla, CA
blhanson@ucsd.edu

2024 AIAA/AAS Space Flight Mechanics Meeting
Monday 8 January 2024

A Look Back at the History of Orbital State Estimation...

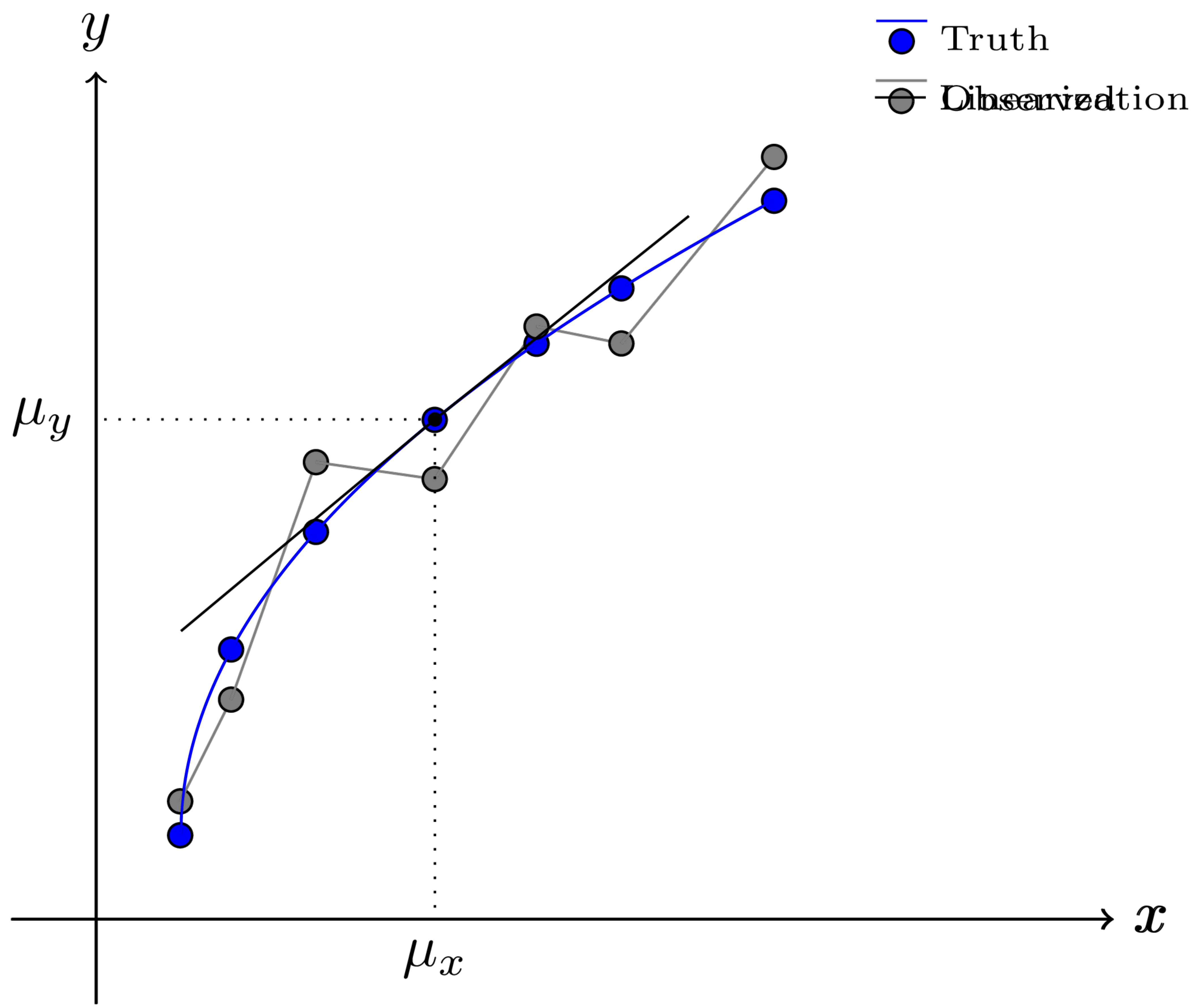


...All Utilized the Extended Kalman Filter!

The Extended Kalman Filter

Assumptions and Limitations

- EKF linearizes the nonlinear system about an estimate mean value, **assuming that the observations are frequent enough that the linearization is accurate**



Model: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{w}(t)$

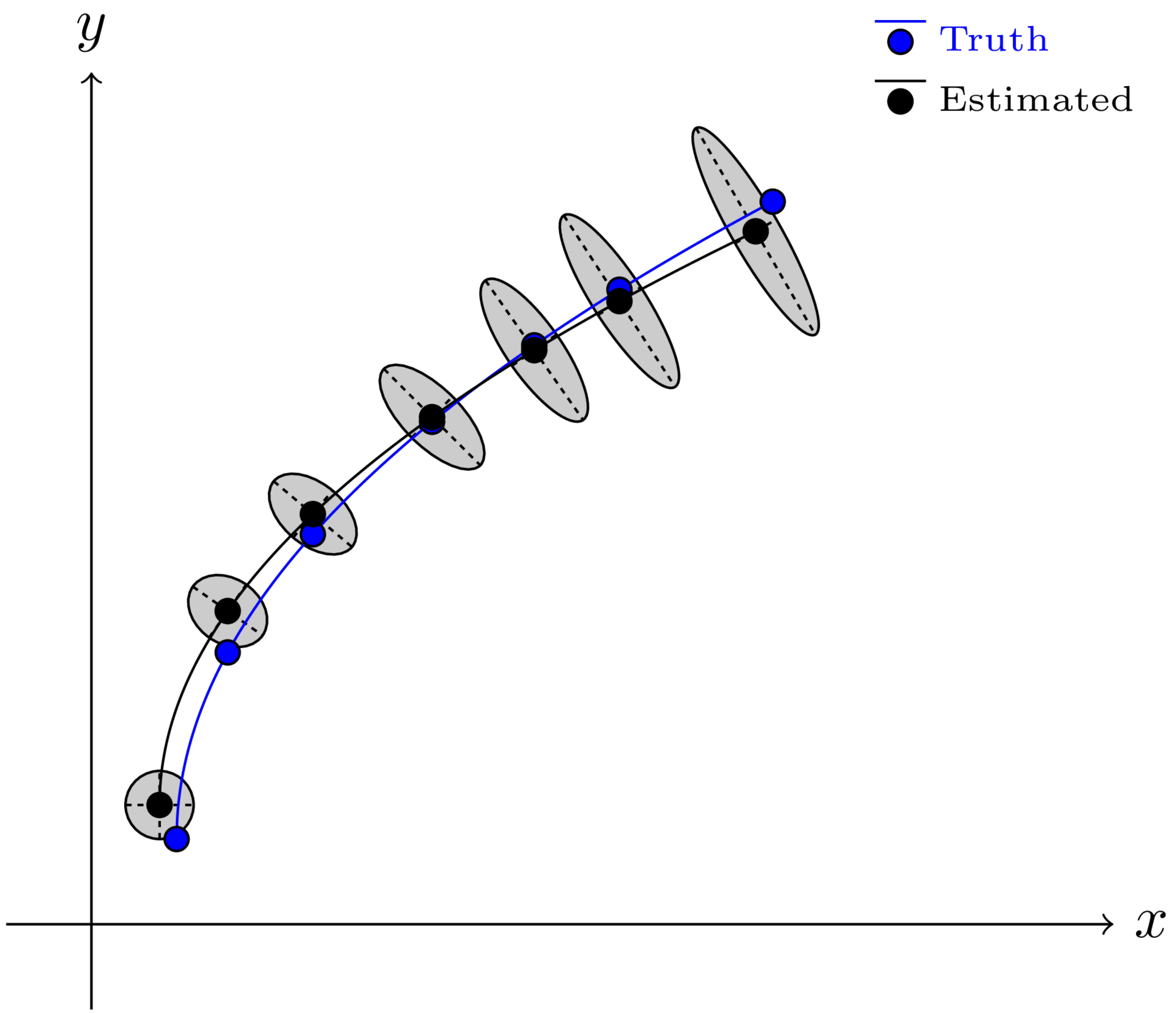
Linearization: $\mathbf{f}(\mathbf{x}, t) \approx \mathbf{f}(\mu, t) + \mathbf{f}'(\mu, t)(\mathbf{x} - \mu)$

Observation: $\mathbf{z} = \mathbf{h}(\mathbf{x}, t) + \mathbf{v}(t)$

The Extended Kalman Filter

Assumptions and Limitations

- EKF linearizes the nonlinear system about an estimate mean value, **assuming that the observations are frequent enough that the linearization is accurate**



Model: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{w}(t)$

Linearization: $\mathbf{f}(\mathbf{x}, t) \approx \mathbf{f}(\mu, t) + \mathbf{f}'(\mu, t)(\mathbf{x} - \mu)$

Observation: $\mathbf{z} = \mathbf{h}(\mathbf{x}, t) + \mathbf{v}(t)$

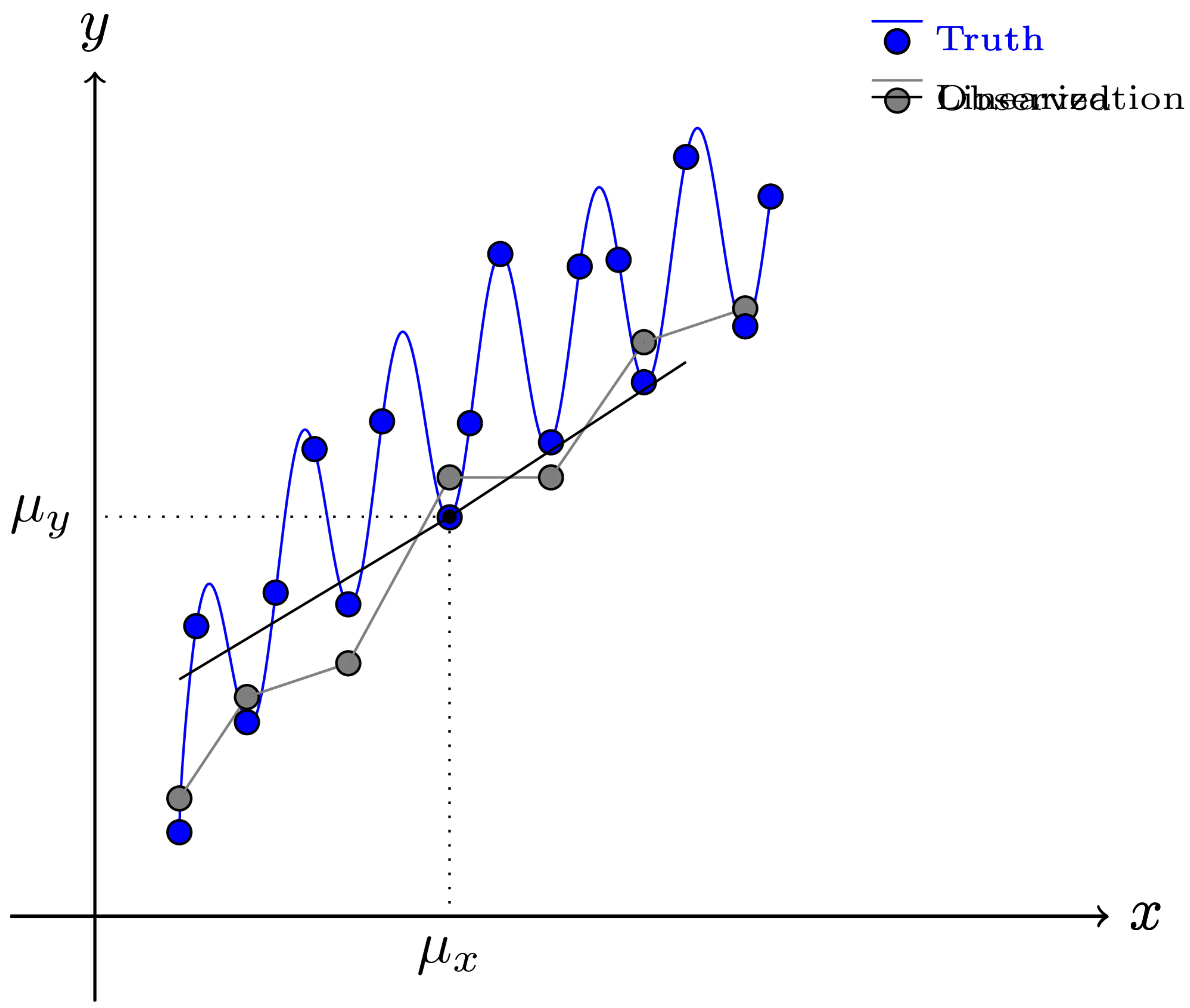
Estimation:

$$\begin{cases} \dot{\hat{\mathbf{x}}} = \mathbf{f}(\hat{\mathbf{x}}, t) + \mathbf{K}(t)[\mathbf{z} - \mathbf{h}(\hat{\mathbf{x}})] \\ \dot{\mathbf{P}}(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}(t)^T - \mathbf{K}(t)\mathbf{H}(t)\mathbf{P}(t) + \mathbf{Q}(t) \\ \mathbf{K}(t) = \mathbf{P}(t)\mathbf{H}(t)^T\mathbf{R}(t)^{-1} \\ \mathbf{F}(t) = \left.\dfrac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\hat{\mathbf{x}}}, \quad \mathbf{H}(t) = \left.\dfrac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\hat{\mathbf{x}}} \end{cases}$$

The Extended Kalman Filter

Assumptions and Limitations

- EKF linearizes the nonlinear system about an estimate mean value, **assuming that the observations are frequent enough that the linearization is accurate**



Model: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{w}(t)$

Linearization: $\mathbf{f}(\mathbf{x}, t) \approx \mathbf{f}(\mu, t) + \mathbf{f}'(\mu, t)(\mathbf{x} - \mu)$

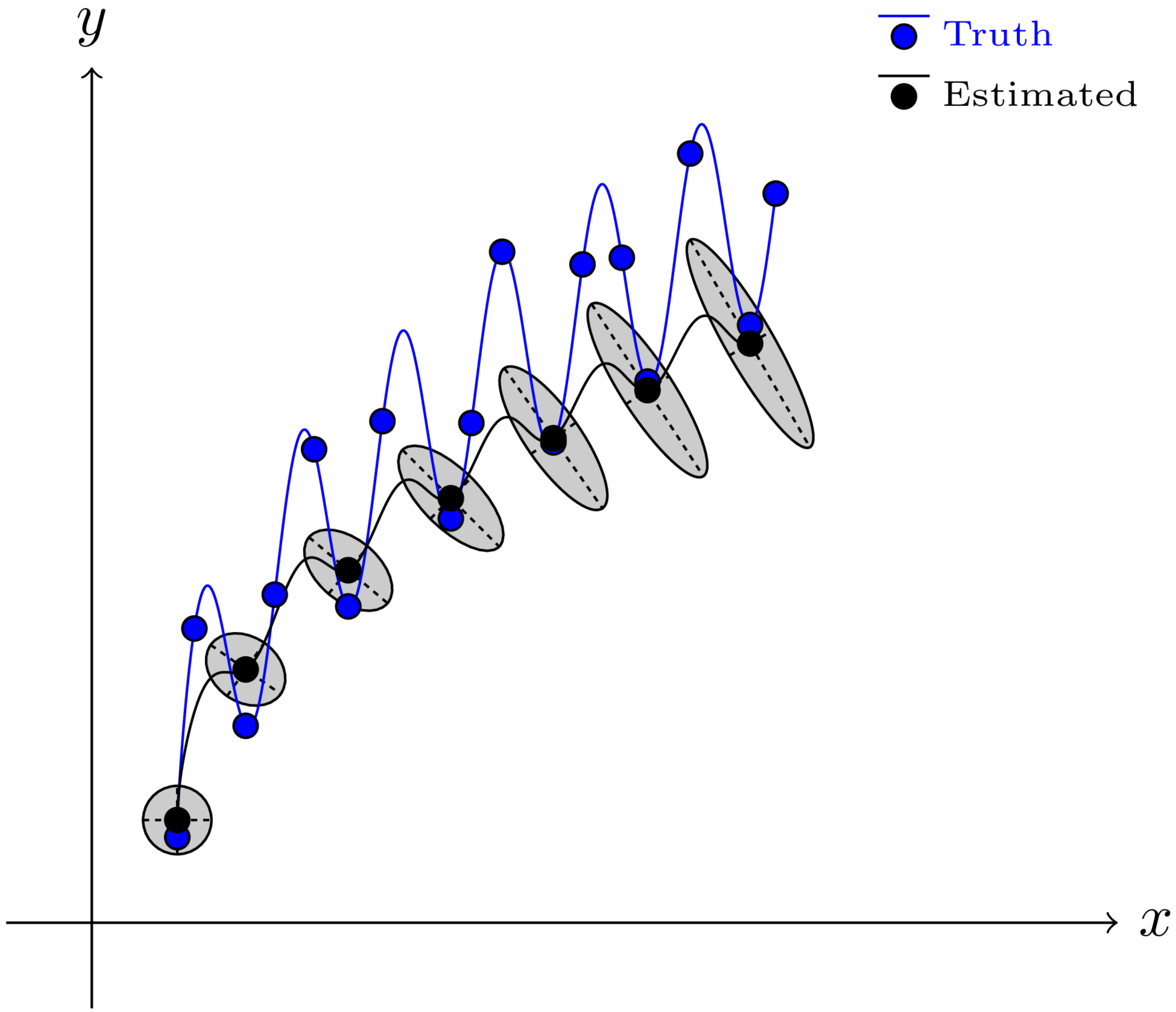
Observation: $\mathbf{z} = \mathbf{h}(\mathbf{x}, t) + \mathbf{v}(t)$

Estimation:
$$\begin{cases} \dot{\hat{\mathbf{x}}} = \mathbf{f}(\hat{\mathbf{x}}, t) + \mathbf{K}(t)[\mathbf{z} - \mathbf{h}(\hat{\mathbf{x}})] \\ \dot{\mathbf{P}}(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}(t)^T - \mathbf{K}(t)\mathbf{H}(t)\mathbf{P}(t) + \mathbf{Q}(t) \\ \mathbf{K}(t) = \mathbf{P}(t)\mathbf{H}(t)^T\mathbf{R}(t)^{-1} \\ \mathbf{F}(t) = \left.\dfrac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\hat{\mathbf{x}}}, \quad \mathbf{H}(t) = \left.\dfrac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\hat{\mathbf{x}}} \end{cases}$$

The Extended Kalman Filter

Assumptions and Limitations

- EKF linearizes the nonlinear system about an estimate mean value, **assuming that the observations are frequent enough that the linearization is accurate**



Model: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{w}(t)$

Linearization: $\mathbf{f}(\mathbf{x}, t) \approx \mathbf{f}(\mu, t) + \mathbf{f}'(\mu, t)(\mathbf{x} - \mu)$

Observation: $\mathbf{z} = \mathbf{h}(\mathbf{x}, t) + \mathbf{v}(t)$

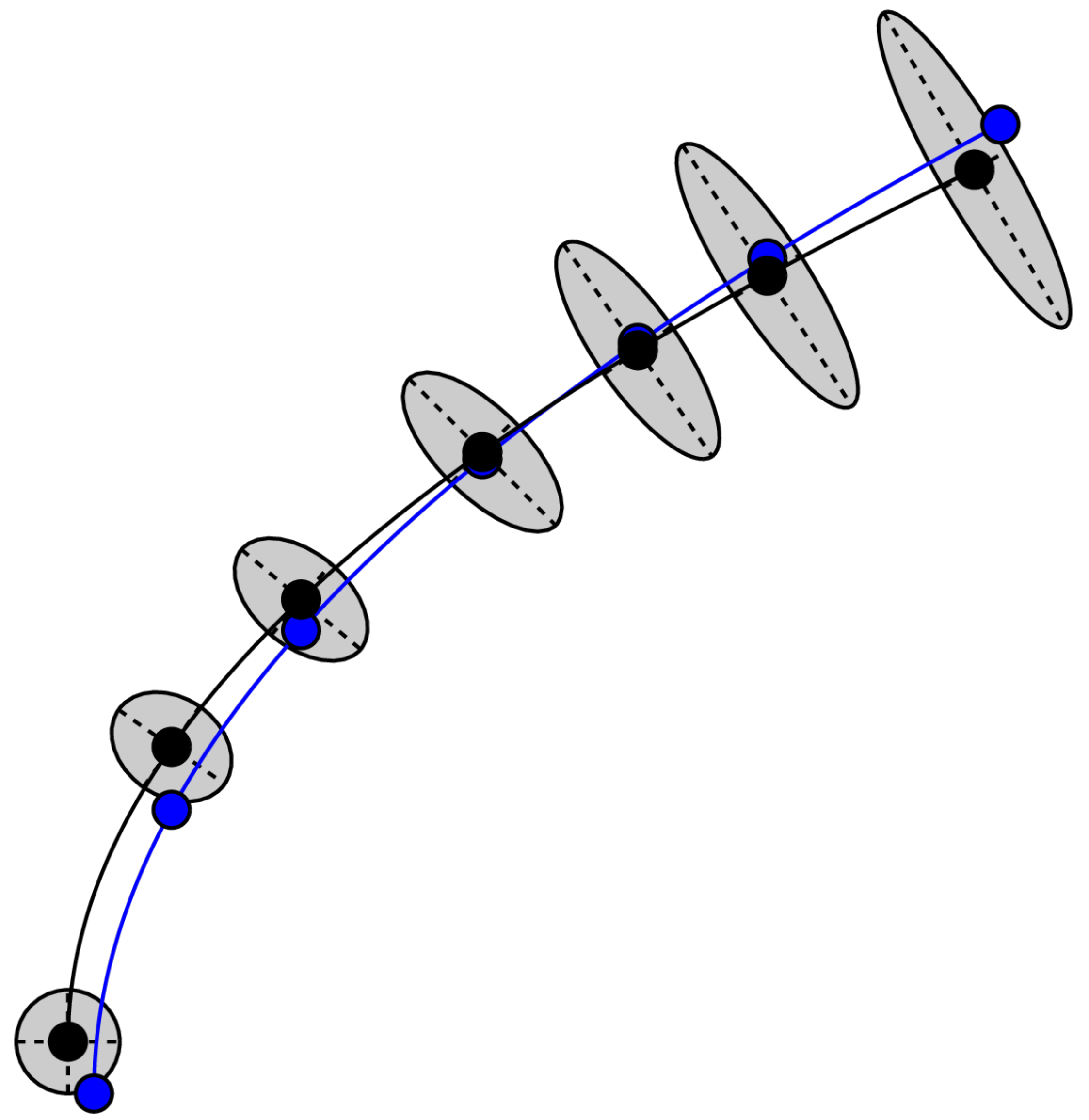
Estimation:
$$\begin{cases} \dot{\hat{\mathbf{x}}} = \mathbf{f}(\hat{\mathbf{x}}, t) + \mathbf{K}(t)[\mathbf{z} - \mathbf{h}(\hat{\mathbf{x}})] \\ \dot{\mathbf{P}}(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}(t)^T - \mathbf{K}(t)\mathbf{H}(t)\mathbf{P}(t) + \mathbf{Q}(t) \\ \mathbf{K}(t) = \mathbf{P}(t)\mathbf{H}(t)^T\mathbf{R}(t)^{-1} \\ \mathbf{F}(t) = \left.\dfrac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\hat{\mathbf{x}}}, \quad \mathbf{H}(t) = \left.\dfrac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\hat{\mathbf{x}}} \end{cases}$$

- For the majority of past space missions, the observation frequency has made linearization error negligible - **this may not always be the case!**

Current Landscape of Uncertainty Propagation Methods

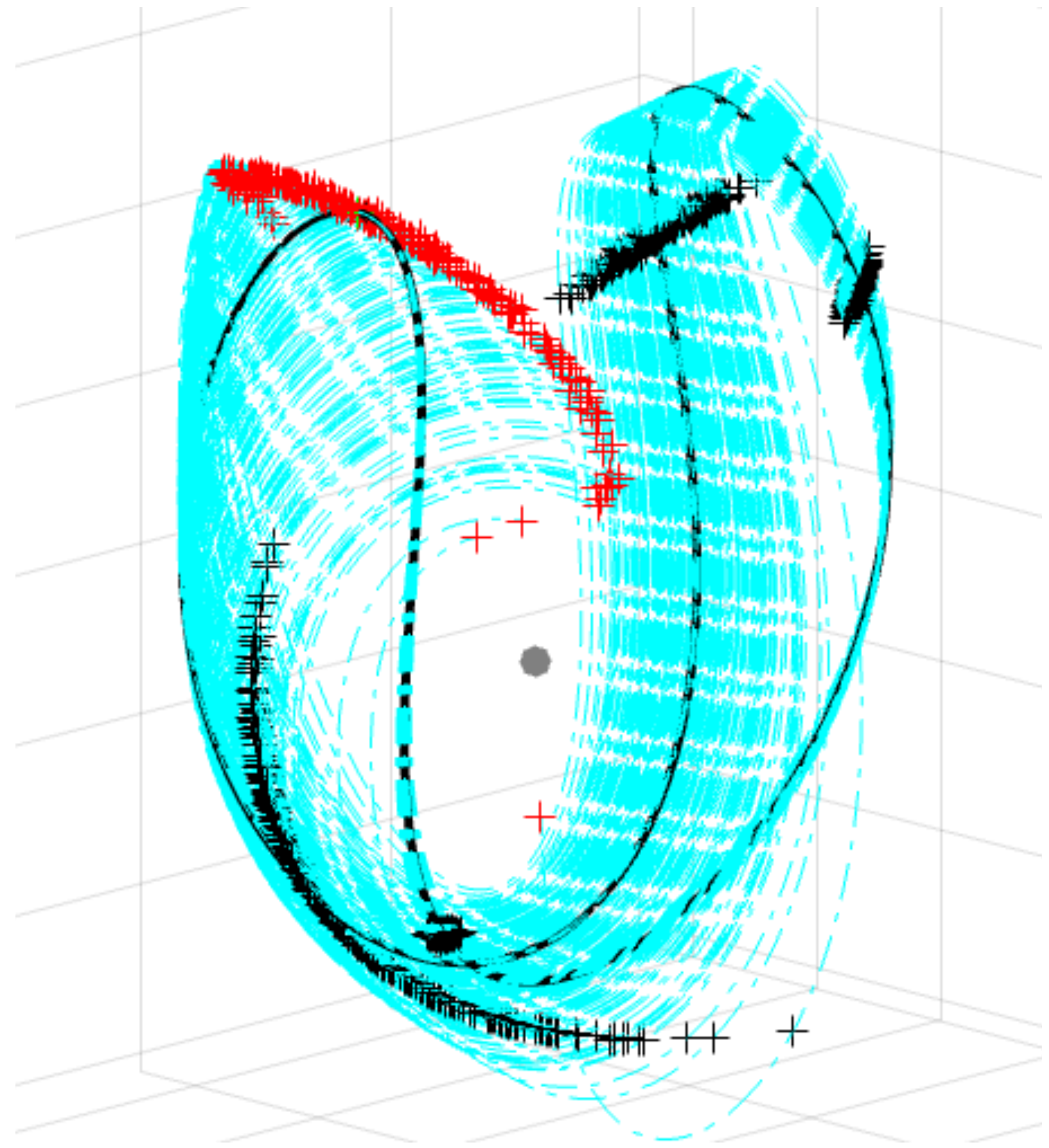
Alternatives to the Extended Kalman Filter

Kalman Filters Variations



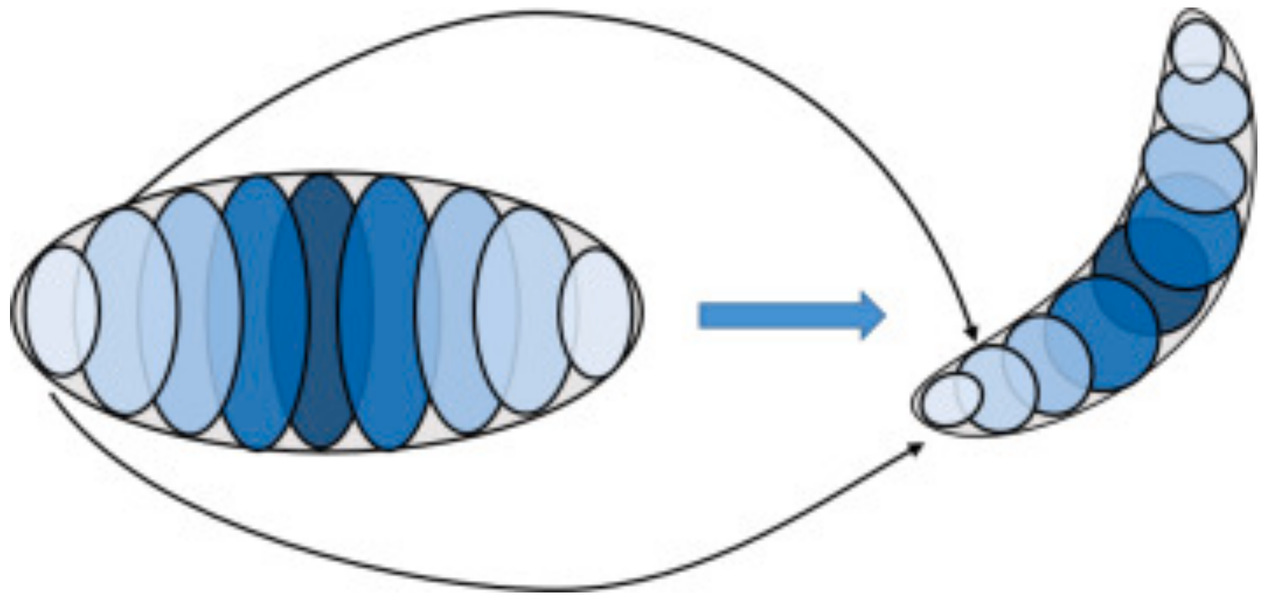
- Accuracy is predicated on frequent measurement updates such that linearization errors are negligible
- Extremely chaotic dynamics means the measurement frequency requirement may not be feasible for certain regimes

Monte Carlo/Particle Filters



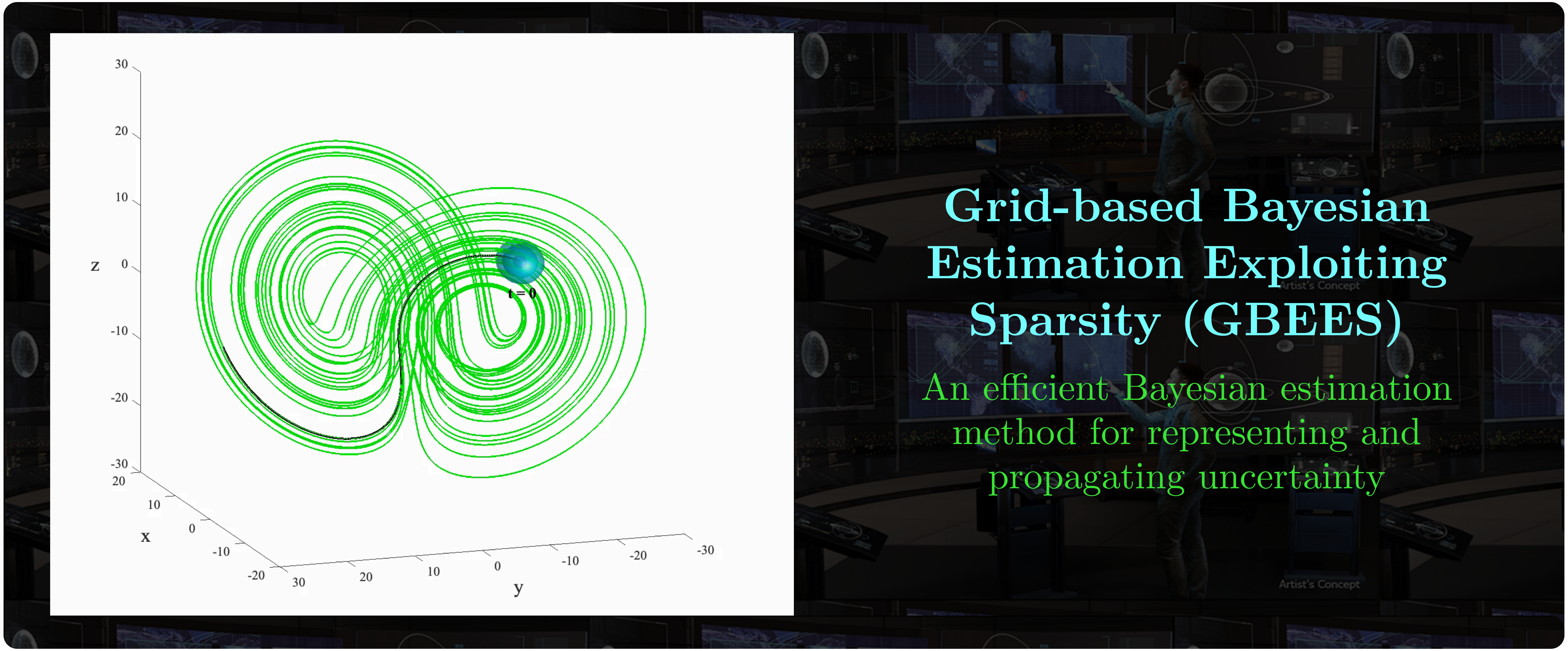
- "Monte Carlo has some core fundamental limitations… the computational burden becomes prohibitive for very low probability events." - NASA Technology Roadmaps, 2015
- A large, **static** number of particles are required to represent non-Gaussian uncertainty

Gaussian Mixture Models



- Depicts a non-Gaussian uncertainty as a mixture of Gaussian distributions
- Representing highly non-Gaussian distributions requires a large number of Gaussian distributions, equating to an ad-hoc splitting/weighting procedure and a computational burden

- To represent state uncertainty in chaotic regimes, novel uncertainty propagation methods must be accurate for long periods of time in the absence of measurement updates, represent non-Gaussian distributions, consider epistemic uncertainty, and be scalable to high-dimensional problems.



General Formulation

Mixed Discrete/Continuous Propagation

- GBEES consists of two distinct processes, one performed in **continuous-time**, the other in **discrete-time**:

1. The probability distribution function $p_{\mathbf{x}}(\mathbf{x}', t)$ is continuous-time marched via the **Fokker-Planck Equation**:

$$\frac{\partial p_{\mathbf{x}}(\mathbf{x}', t)}{\partial t} = -\frac{\partial f_i(\mathbf{x}', t) p_{\mathbf{x}}(\mathbf{x}', t)}{\partial x_i'} + \frac{1}{2}\frac{\partial^2 q_{ij} p_{\mathbf{x}}(\mathbf{x}', t)}{\partial x_i' \partial x_j'}$$

   - f_i: advection (EOMs) in the i^{th} dimension
   - q_{ij}: $(i, j)^{\text{th}}$ element of the spectral density ($Q = 0$, PDE is hyperbolic)

2. At discrete-time interval t_k, measurement y_k updates $p_{\mathbf{x}}(\mathbf{x}', t)$ via **Bayes' Theorem**:

$$p_{\mathbf{x}}(\mathbf{x}', t_{k+}) = \frac{p_{\mathbf{y}}(\mathbf{y}_k \,|\, \mathbf{x}') p_{\mathbf{x}}(\mathbf{x}', t_{k-})}{C}$$

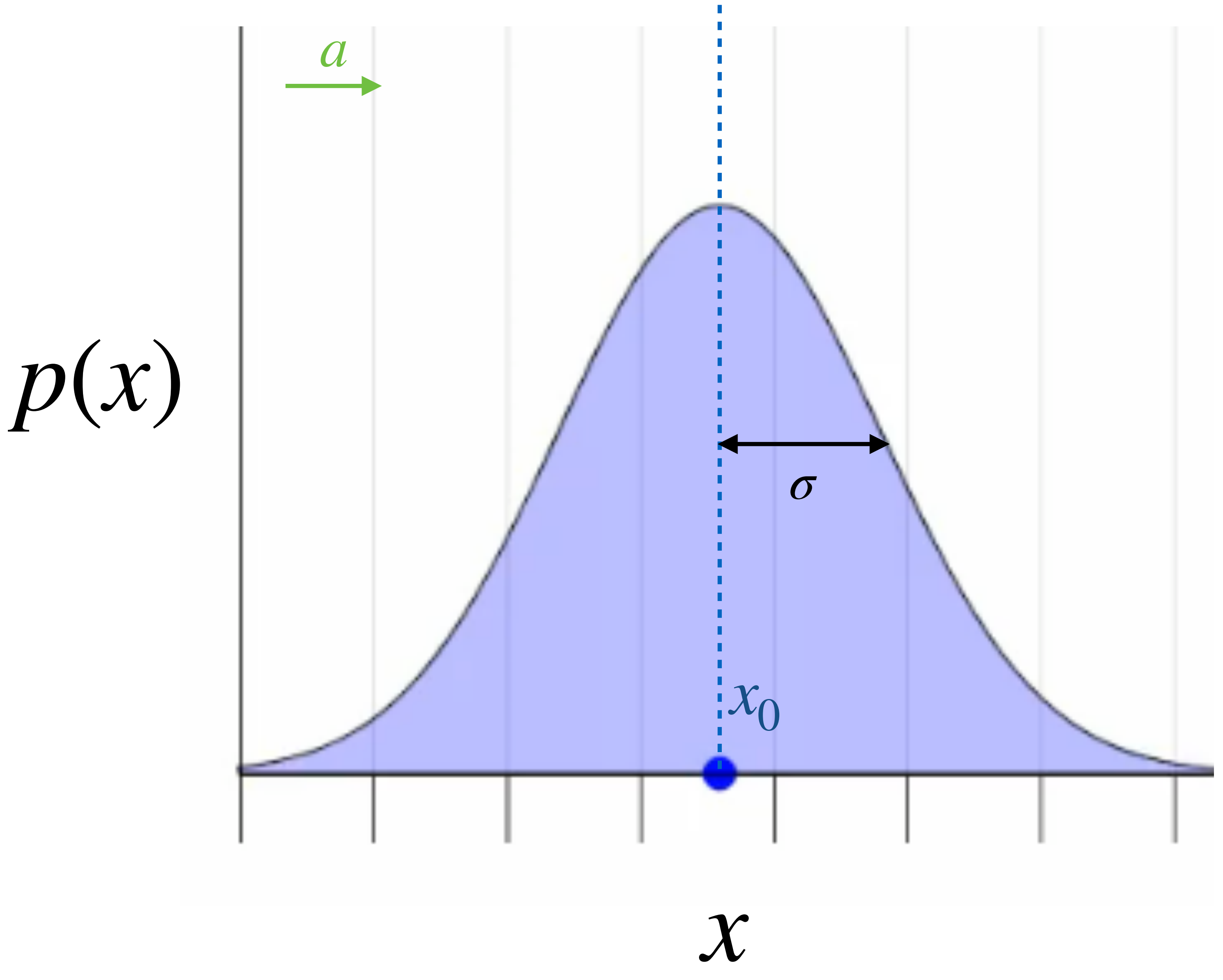
   - $p_{\mathbf{x}}(\mathbf{x}', t_{k+})$: a posteriori distribution
   - $p_{\mathbf{y}}(\mathbf{y}_k \,|\, \mathbf{x}')$: measurement distribution
   - $p_{\mathbf{x}}(\mathbf{x}', t_{k-})$: a priori distribution
   - C: normalization constant

Grid-based Bayesian Estimation Exploiting Sparsity (GBEES)

- Consider a 1-dimensional, linear test example:

$$\mathbf{x} = [x], \quad \frac{d\mathbf{x}}{dt} = [a], \quad a > 0$$

- Initial observation of $x(t)$ results in a Gaussian PDF $p(x)$ centered about x_0 with standard deviation σ

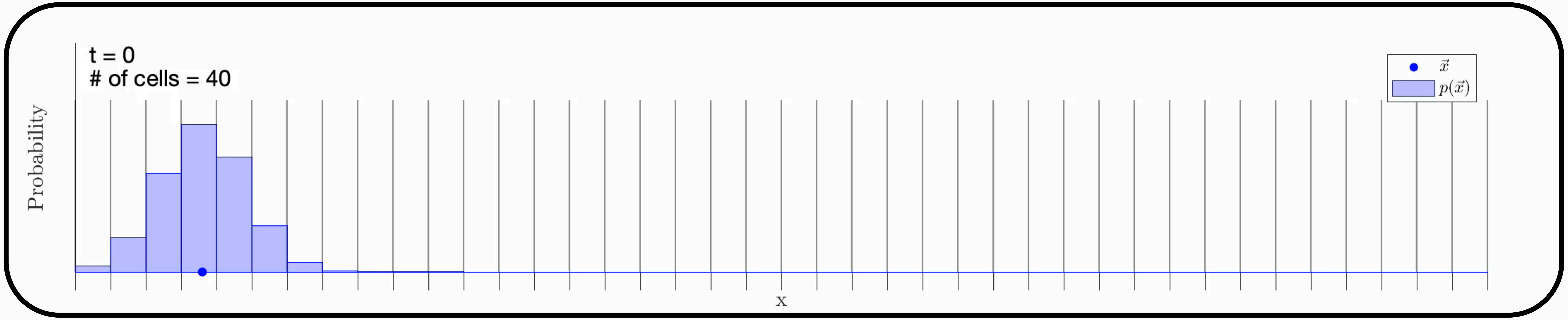


- How does $p(x)$, governed by $d\mathbf{x}/dt$, change with respect to t?

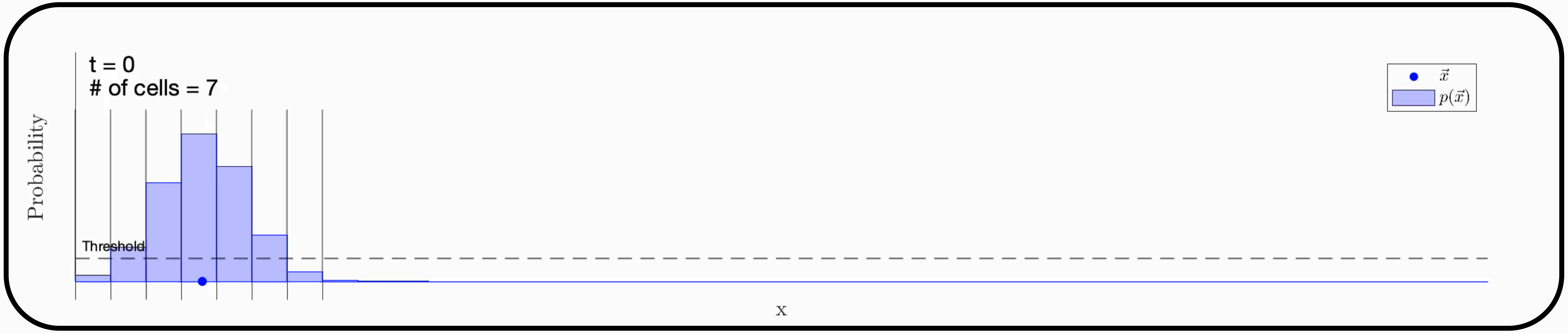
GBEES treats probability as a fluid, and time-evolves the distribution subject to the Fokker-Planck Equation via a Godunov, 2nd order-accurate, finite volume method.

Grid-based Bayesian Estimation Exploiting Sparsity (GBEES)

- **Ignoring sparsity**



- **Exploiting sparsity**



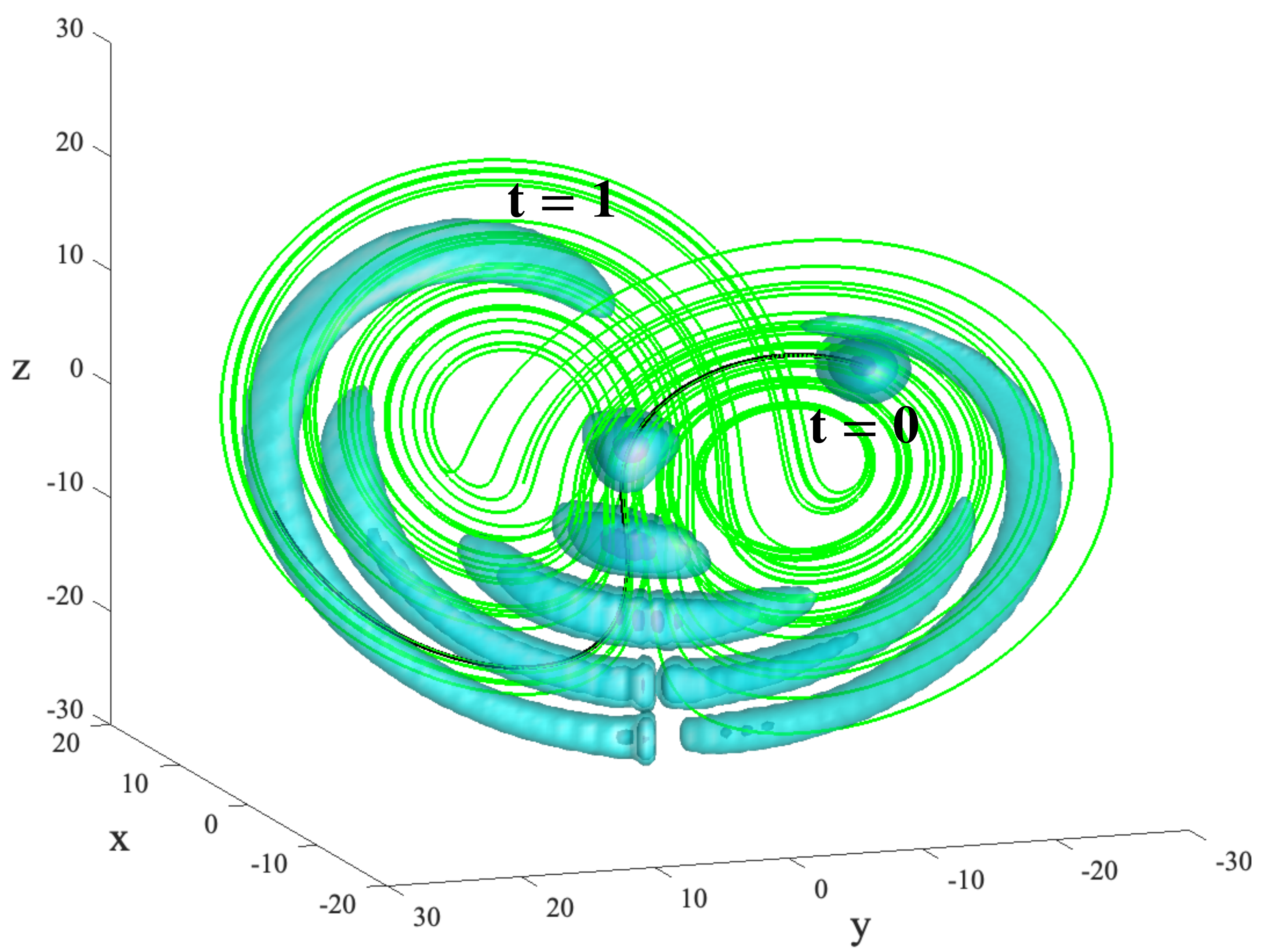
*Not GBEES, just a visual aid

Test Case

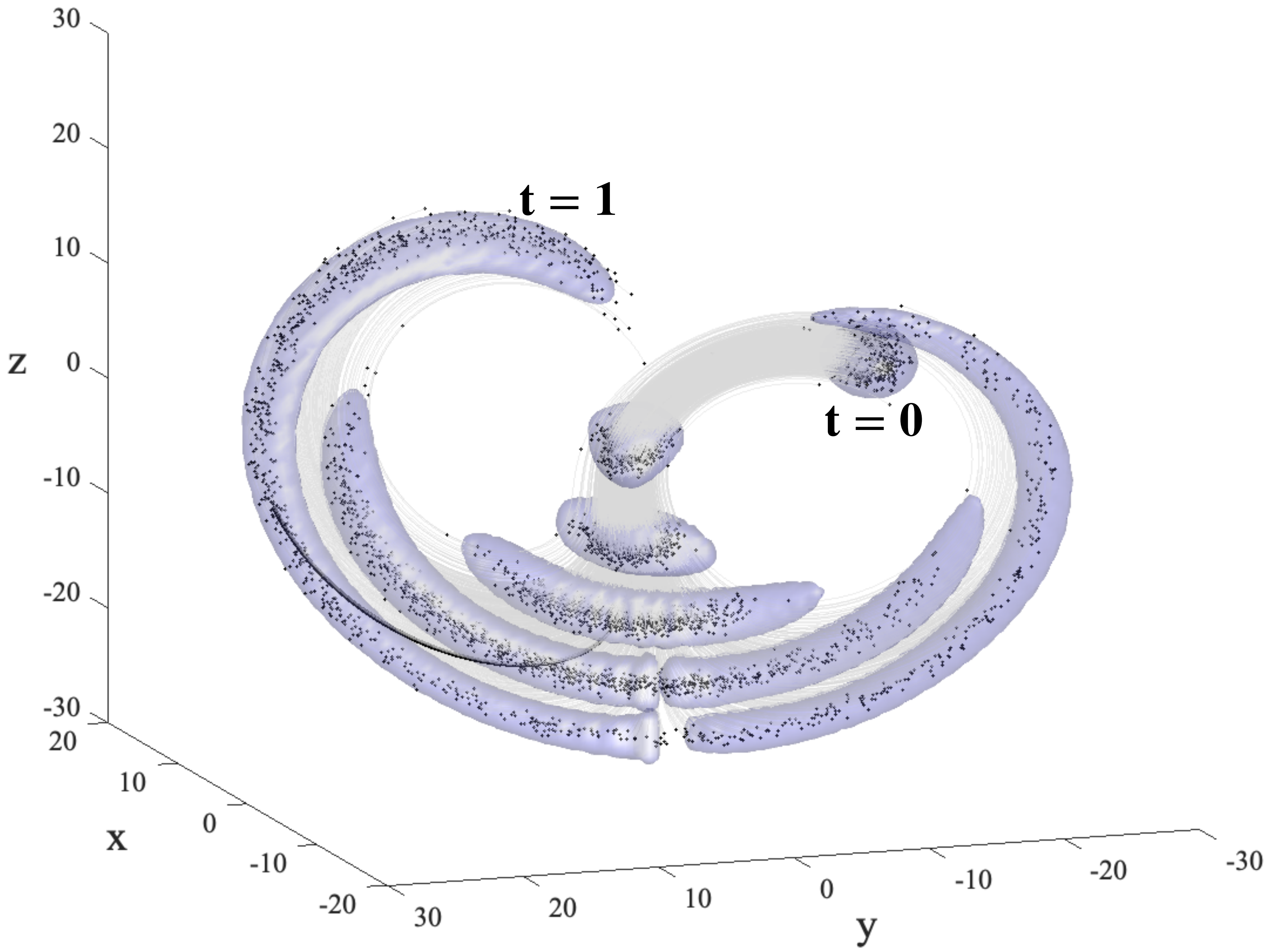
3D Lorenz Attractor

- Colloquially known as the “Butterfly Effect”, the Lorenz attractor is a set of **chaotic** solutions to the Lorenz system
- Test case demonstrates how a Gaussian uncertainty can quickly evolve into a **non-Gaussian uncertainty** (and even split)

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \frac{d\mathbf{x}}{dt} = \begin{bmatrix} \sigma(y-x) \\ -y - xz \\ -b(z+r) - xy \end{bmatrix}$$



$\sigma = 4,\ b = 1,\ r = 48$



GBEES compared with a 500 particle Monte Carlo simulation

Extension to Astrodynamical Problems

Planar Circular Restricted Three-body Problem (PCR3BP)

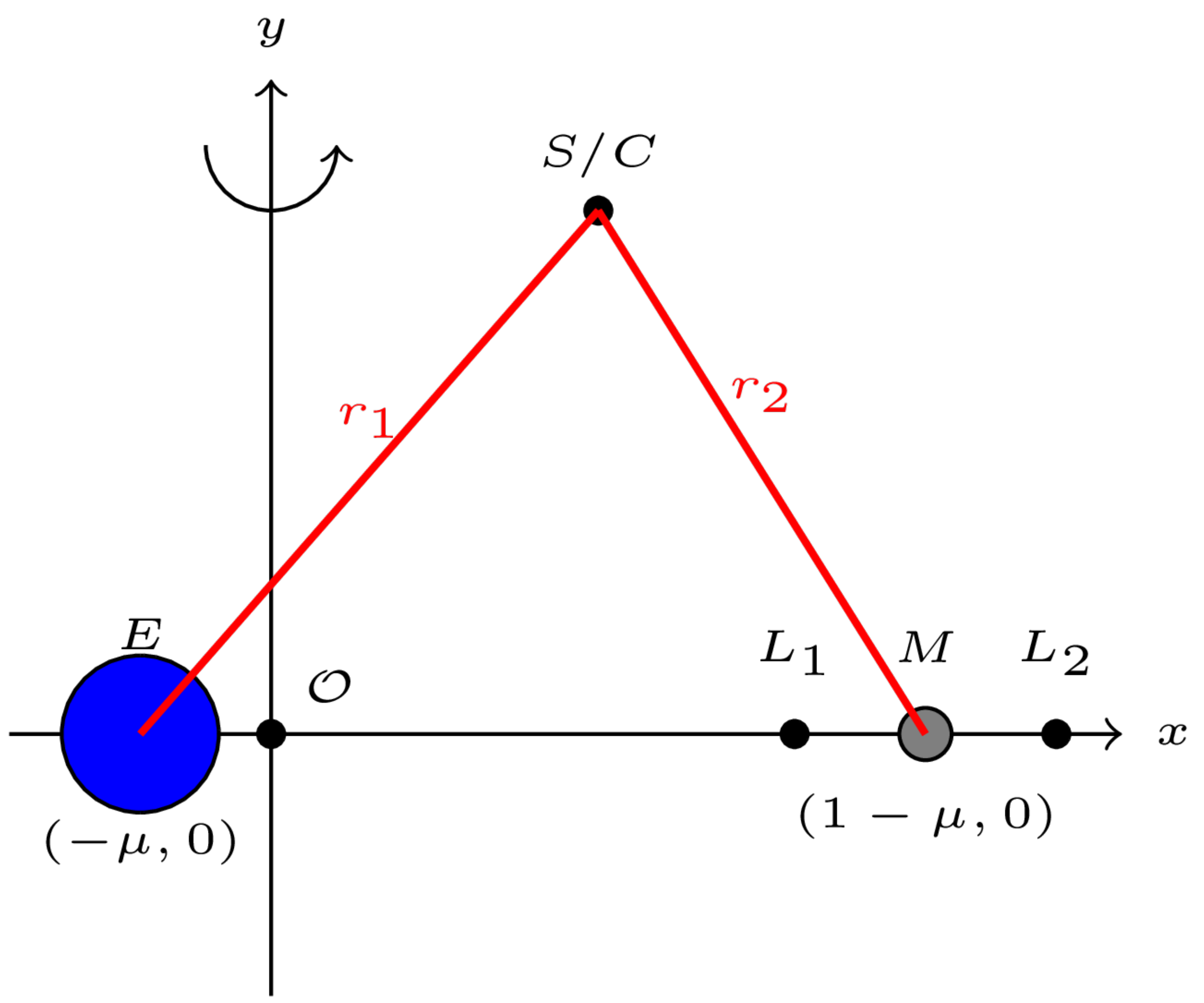
- Jump from low-dimensional, theoretical nonlinear systems to high-dimensional, physical nonlinear systems
- As an astrodynamical test case, we apply GBEES to the planar circular restricted three-body problem (PCR3BP)

PCR3BP

Four-dimensional, nonlinear system

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix}$$

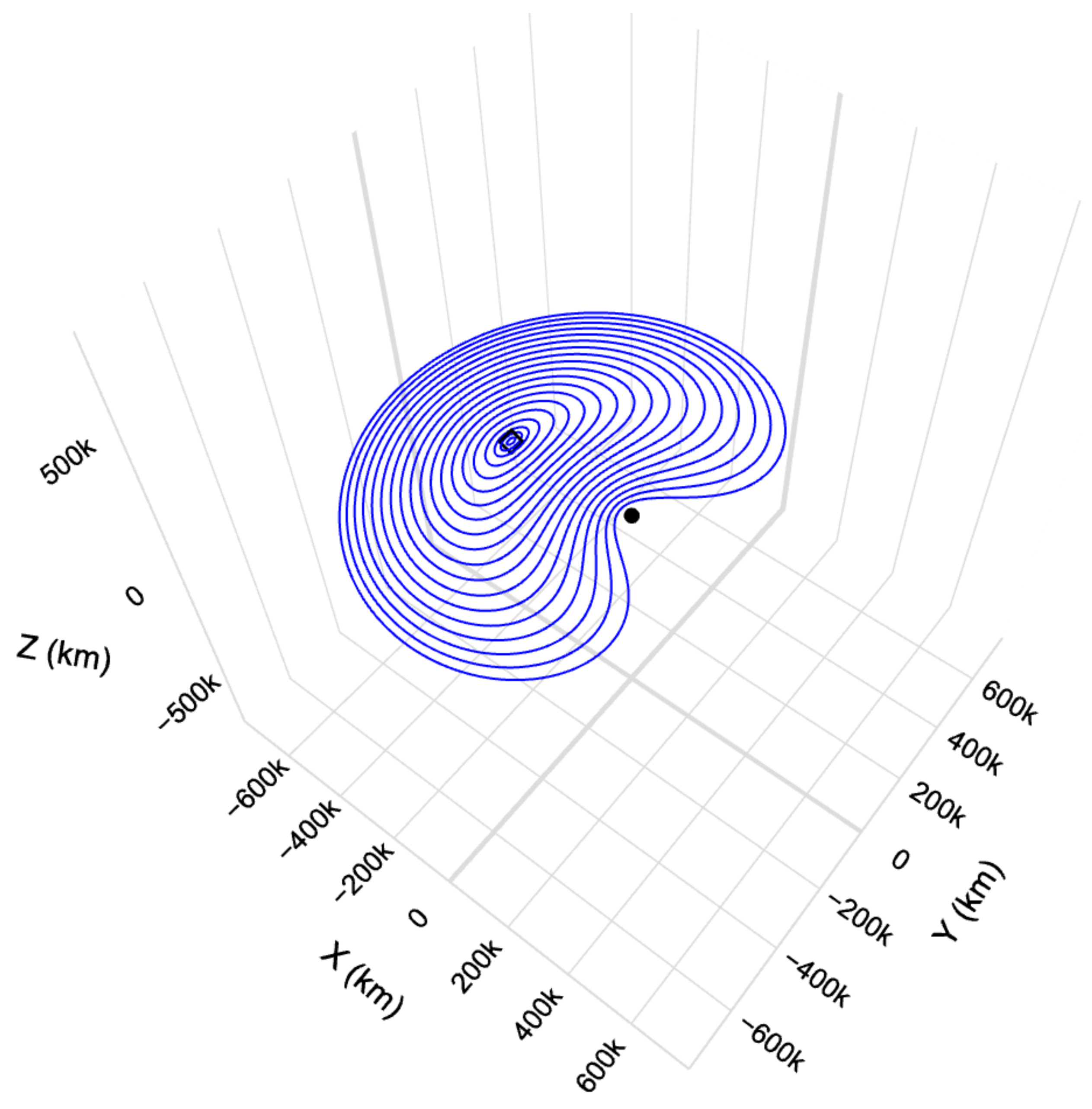
$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} v_x \\ v_y \\ 2v_y + \Omega_x \\ -2v_x + \Omega_y \end{bmatrix}$$



where $\Omega(x, y) = \dfrac{x^2 + y^2}{2} + \dfrac{1-\mu}{r_1} + \dfrac{\mu}{r_2} + \dfrac{\mu(1-\mu)}{2}$

Lyapunov Orbits

Dynamically unstable, entirely planar orbits about the Lagrange points

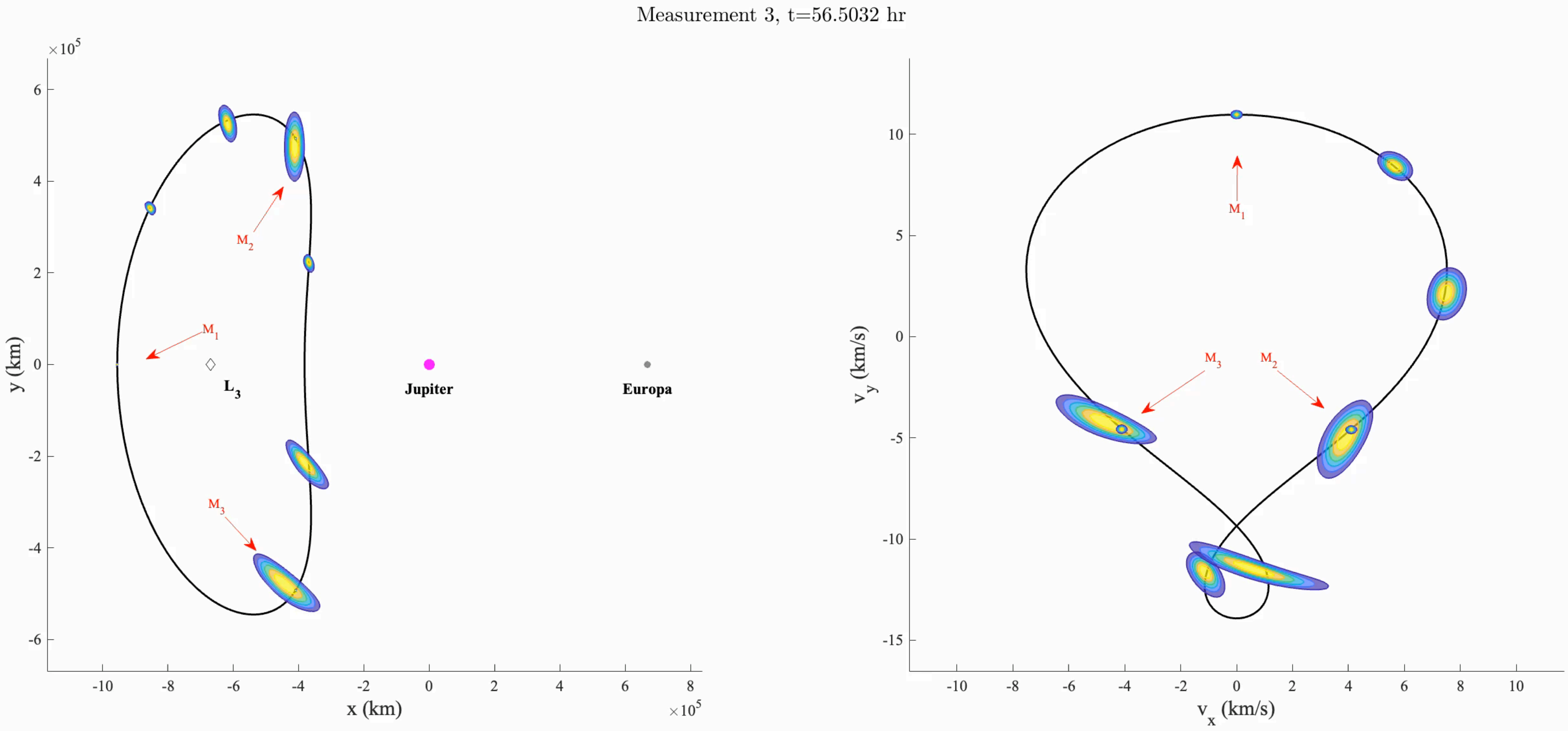


Possible operating trajectories for spacecraft examining celestial systems for multiple orbits - **a great test case for GBEES!**

*Initial conditions generated via JPL Three-Body Periodic Orbit Catalog

Infrequently observed Jupiter-Europa Trajectory

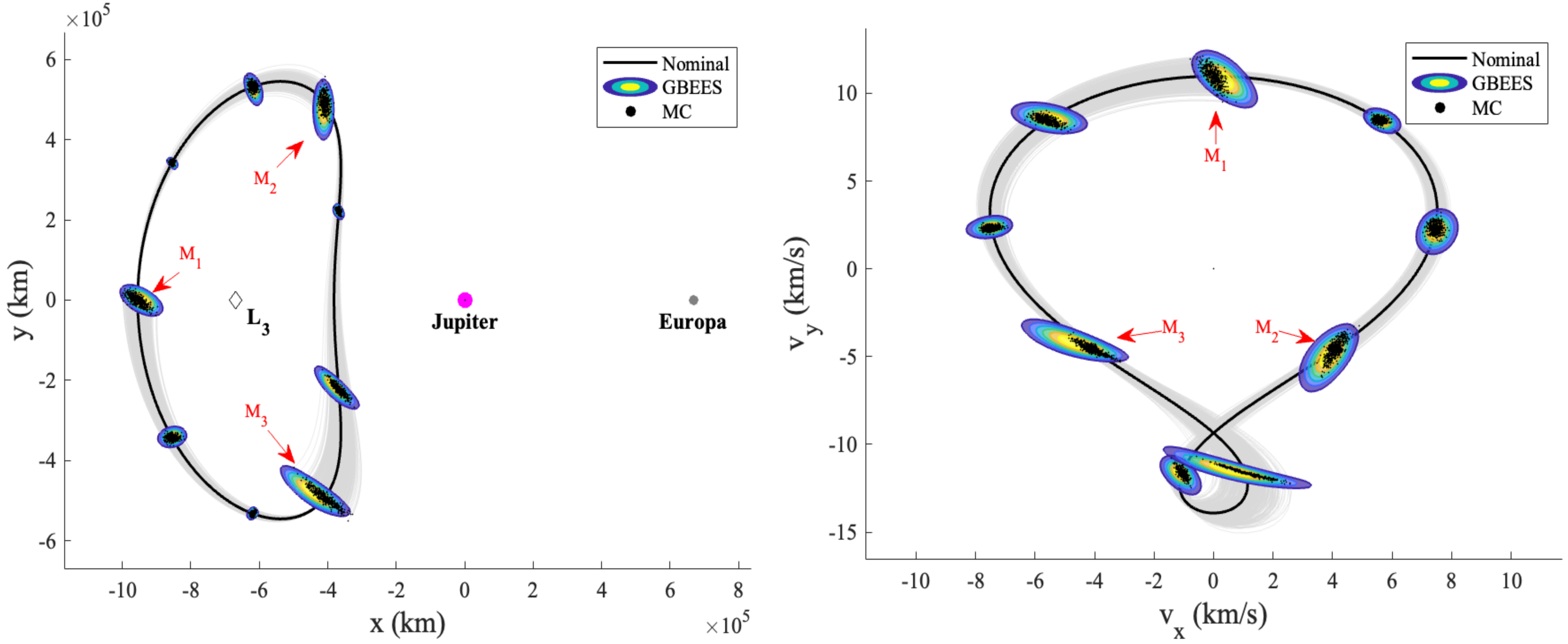
Application of GBEES



Infrequently observed Jupiter-Europa Trajectory

Comparison with MC Simulation - Accuracy

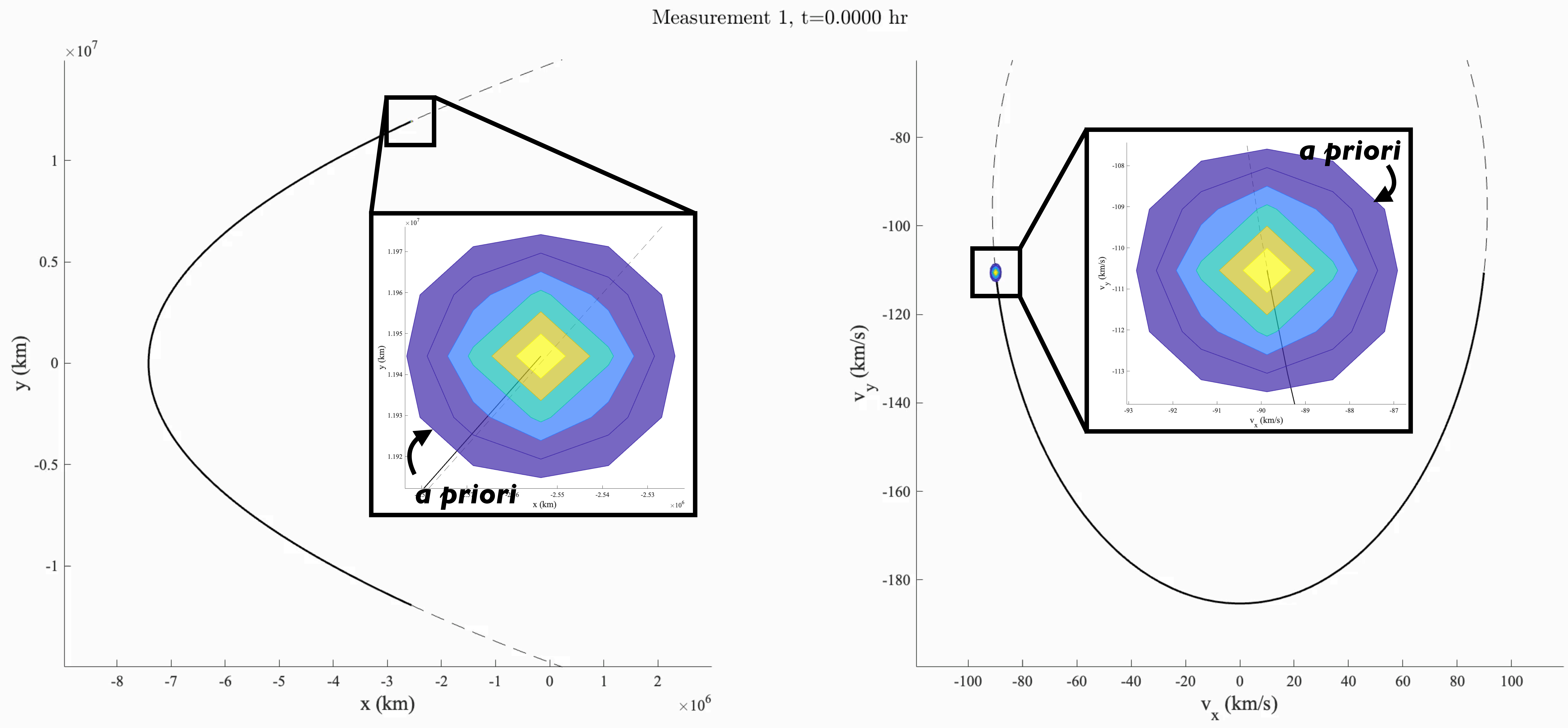
- Comparing results of discretized PDFs propagated via GBEES with 500 particle MC simulation



- **Note:** at discrete measurement intervals M_i, GBEES updates the PDF via Bayes' theorem, while the MC simulation resamples from the new *a priori* measurement, disregarding any prior information

Consideration of epistemic uncertainty for solar probes

Application of GBEES

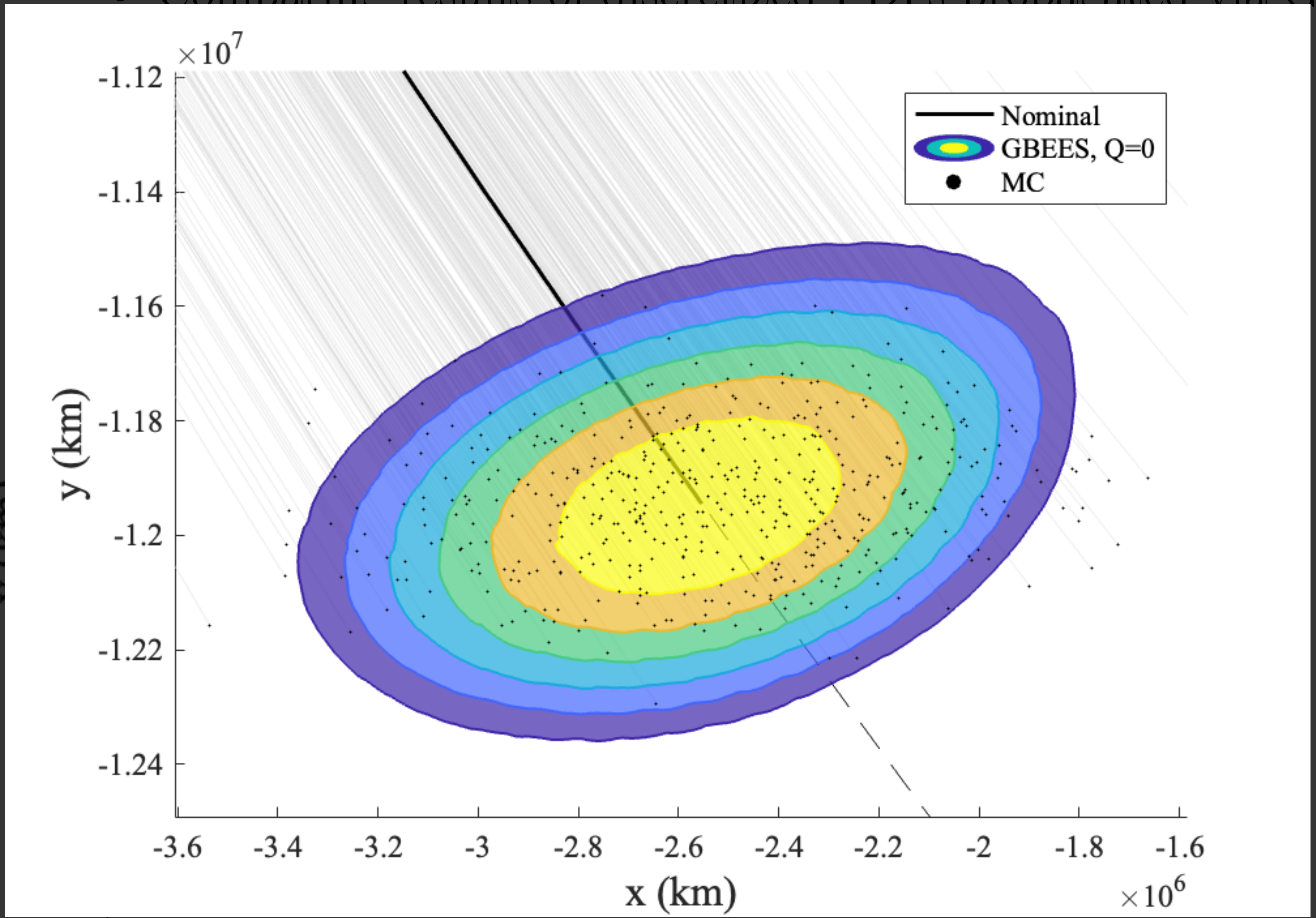


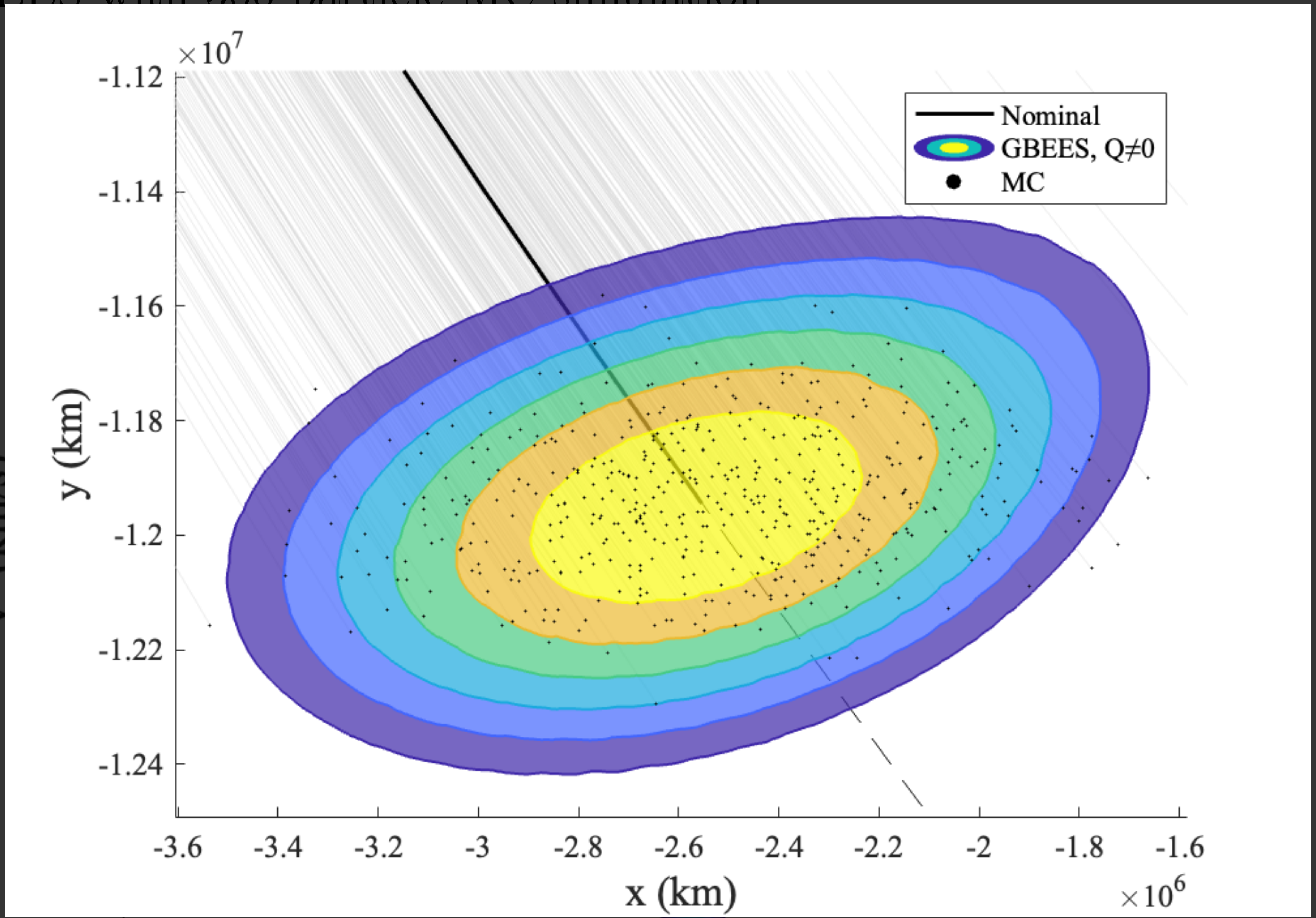
Consideration of epistemic uncertainty for solar probes

Comparison with MC Simulation - Accuracy

- Comparing results of discretized PDFs propagated via GBEES with 500 particle MC simulation

No Diffusion	Diffusion





No Diffusion

- x-position range: $[-3.4136 \times 10^6, \ -1.7636 \times 10^6]$
- y-position range: $[-1.2416 \times 10^7, \ -1.1416 \times 10^7]$

Diffusion

- x-position range: $[-3.5836 \times 10^6, \ -1.5536 \times 10^6]$
- y-position range: $[-1.2536 \times 10^7, \ -1.1406 \times 10^7]$

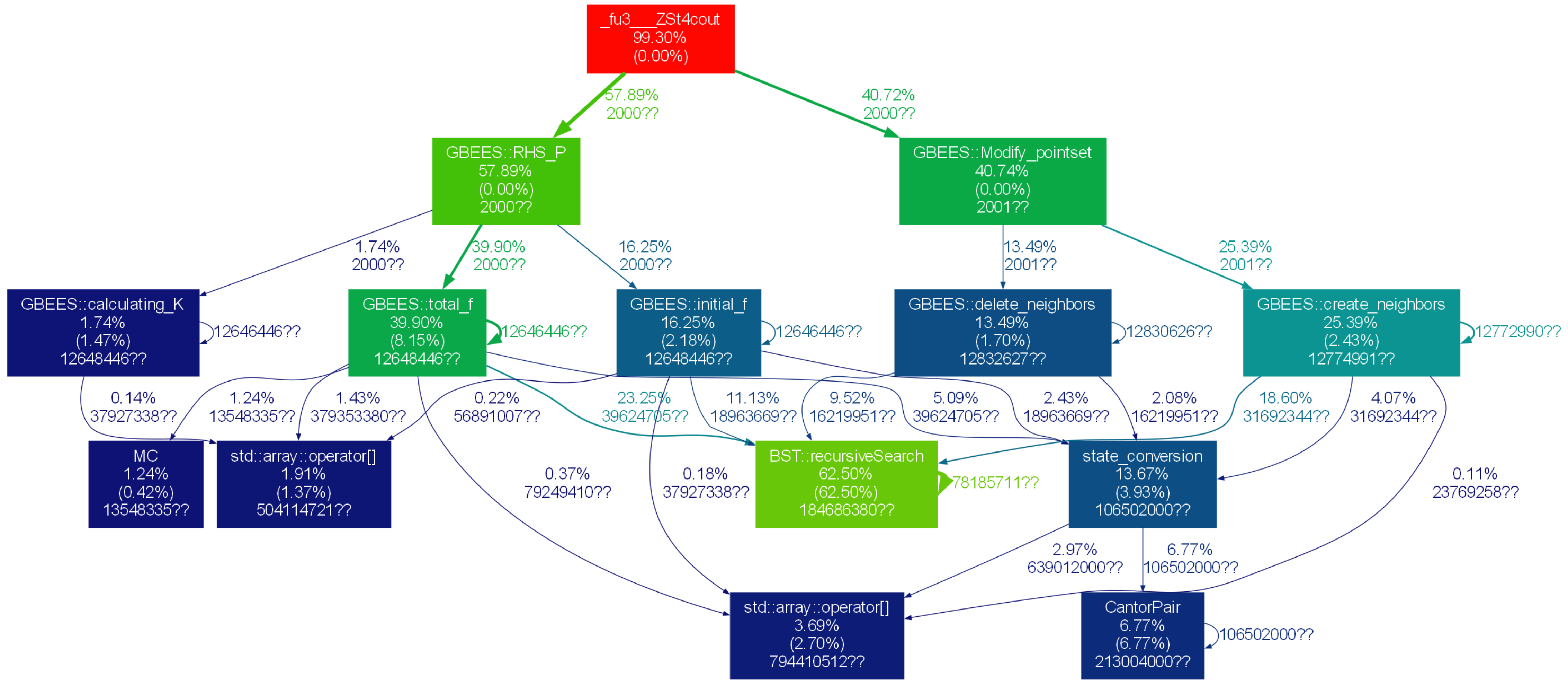
- **Note:** there is no consideration for epistemic uncertainty in the MC simulation

Difference of 380,000 km and 13,000 km in *x*- and *y*-directions, respectively!

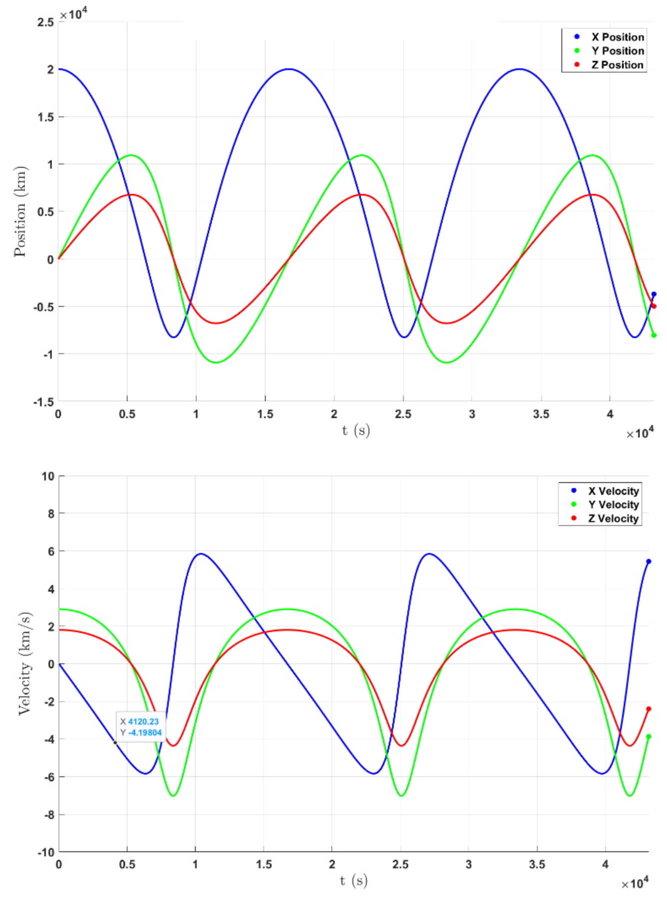
Conclusion

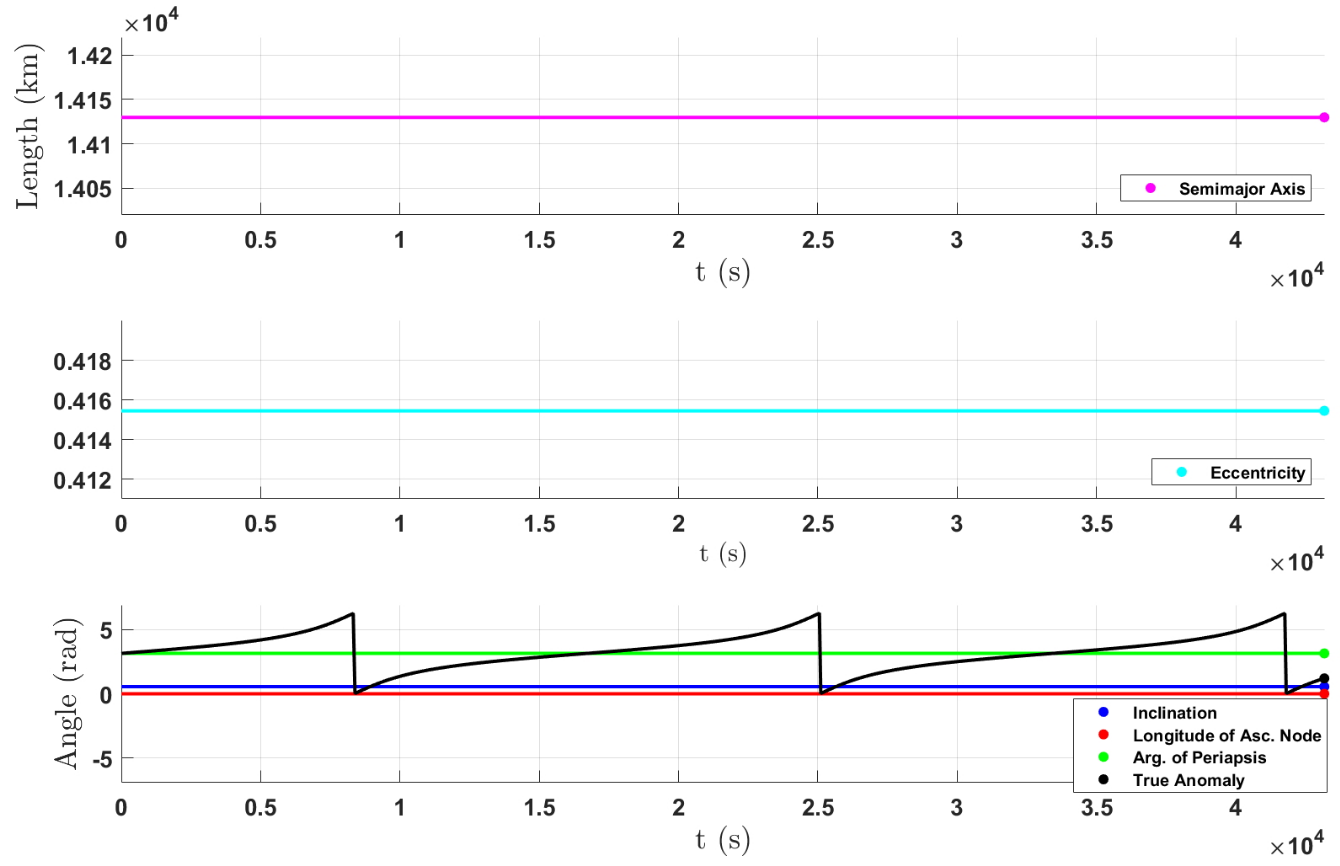
Comments on Results and Future Work

- Accuracy and efficiency results of GBEES compared to MC simulations in regimes where it is expected that the EKF may fail are promising, but require more tweaking if we are to feasibly argue it can compete computationally
  - Rigorous computational profile may determine where bottlenecks are occurring
  - Complete examination of conditions where EKF fail may also be beneficial for demonstrating regimes/mission trajectories of interest



- Next step will be representing and propagating uncertainty in a higher-fidelity, six-dimensional system
  - A jump in dimensionality means an increase in computational burden
  - This may be alleviated by time-marching in a variable set that changes slower (orbital elements)





This investigation was partially supported by the NASA Space Technology Graduate Research Opportunity (Grant Number 80NSSC23K1219)

All code can be found at: https://github.com/bhanson10/GBEES

Thank you for your time. Questions?