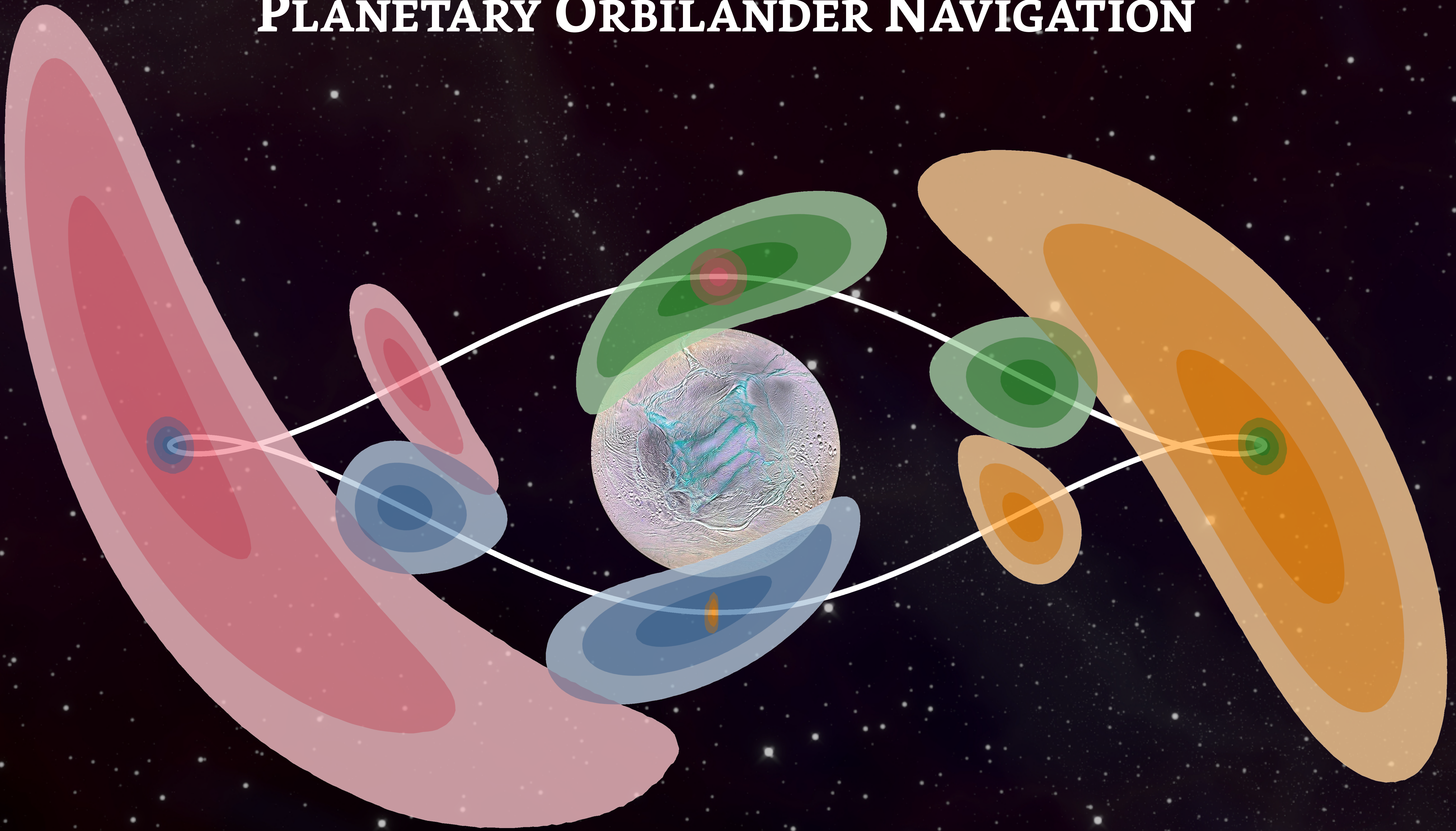
Non-Gaussian Recursive Bayesian Filtering for Outer Planetary Orbilander Navigation



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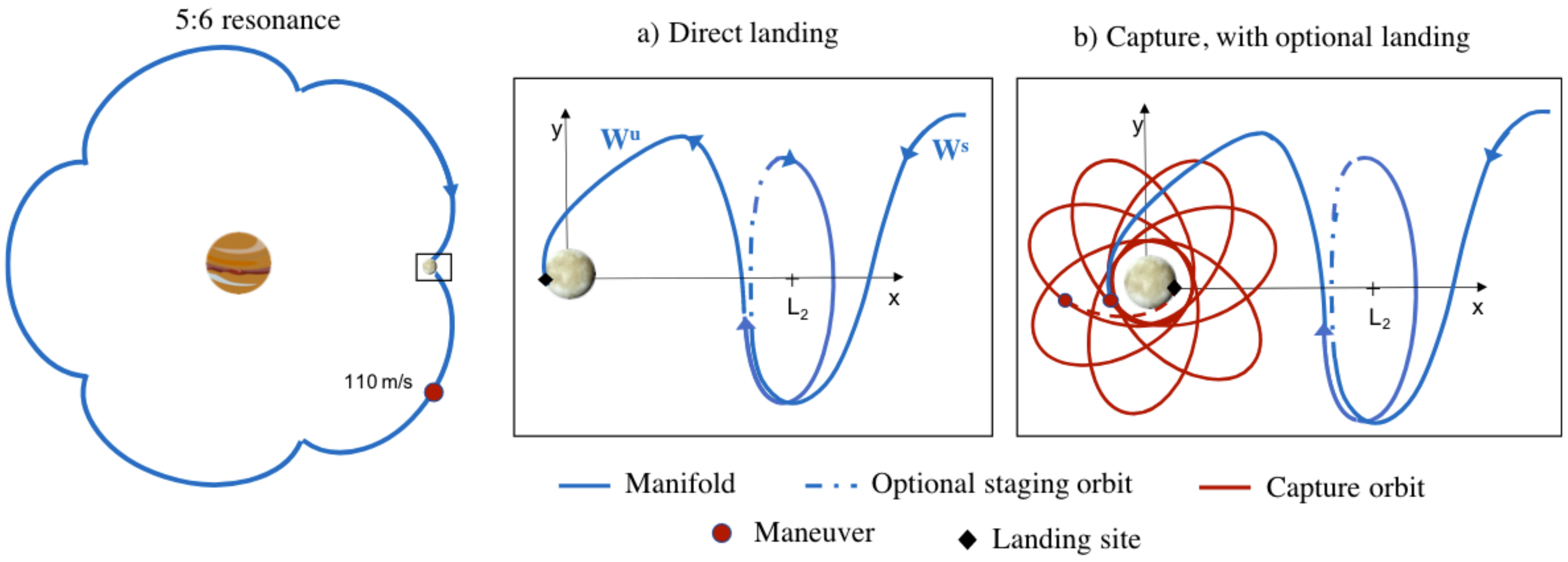
Case Study: Low-Energy Trajectories for Europa Lander

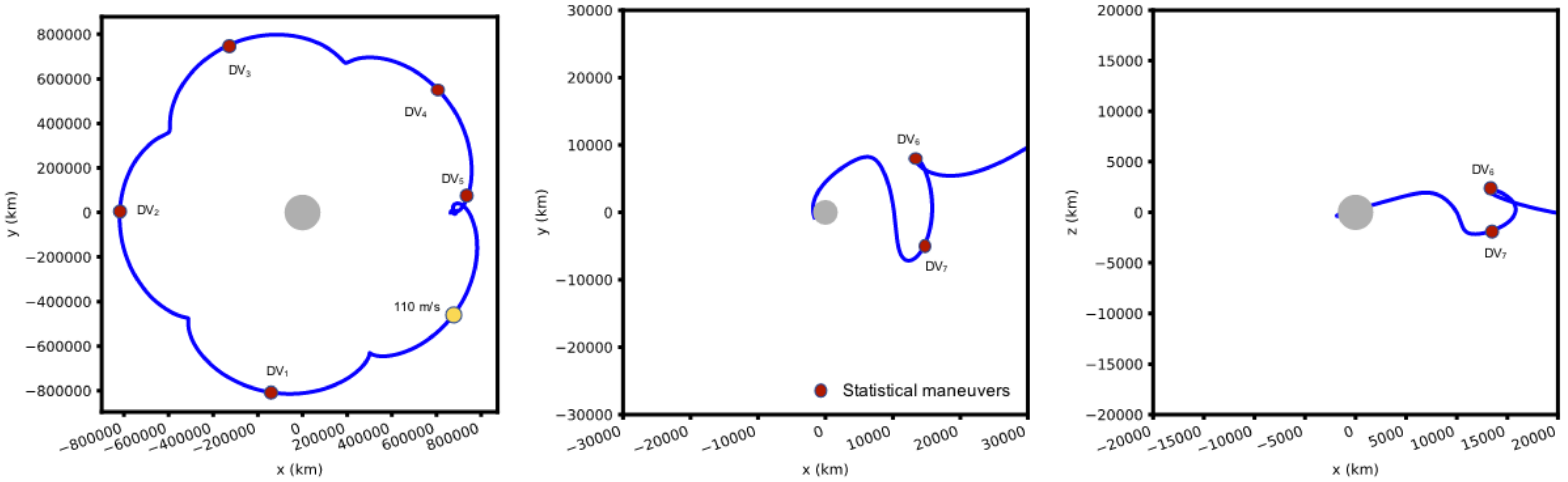
Time validity of the Gaussian assumption of uncertainty

- A theoretically ΔV-free, ballistic capture of a Europa lander realistically requires **statistical maneuvers** to maintain Gaussian error in position and velocity for linearized navigation techniques

Proposed ΔV-free ballistic capture



Actual trajectory with statistical maneuvers ΔV_i



McElrath, Tim, et al. "Navigating low-energy trajectories to land on the surface of Europa." NTRS. (2021)

Case Study: Low-Energy Trajectories for Europa Lander

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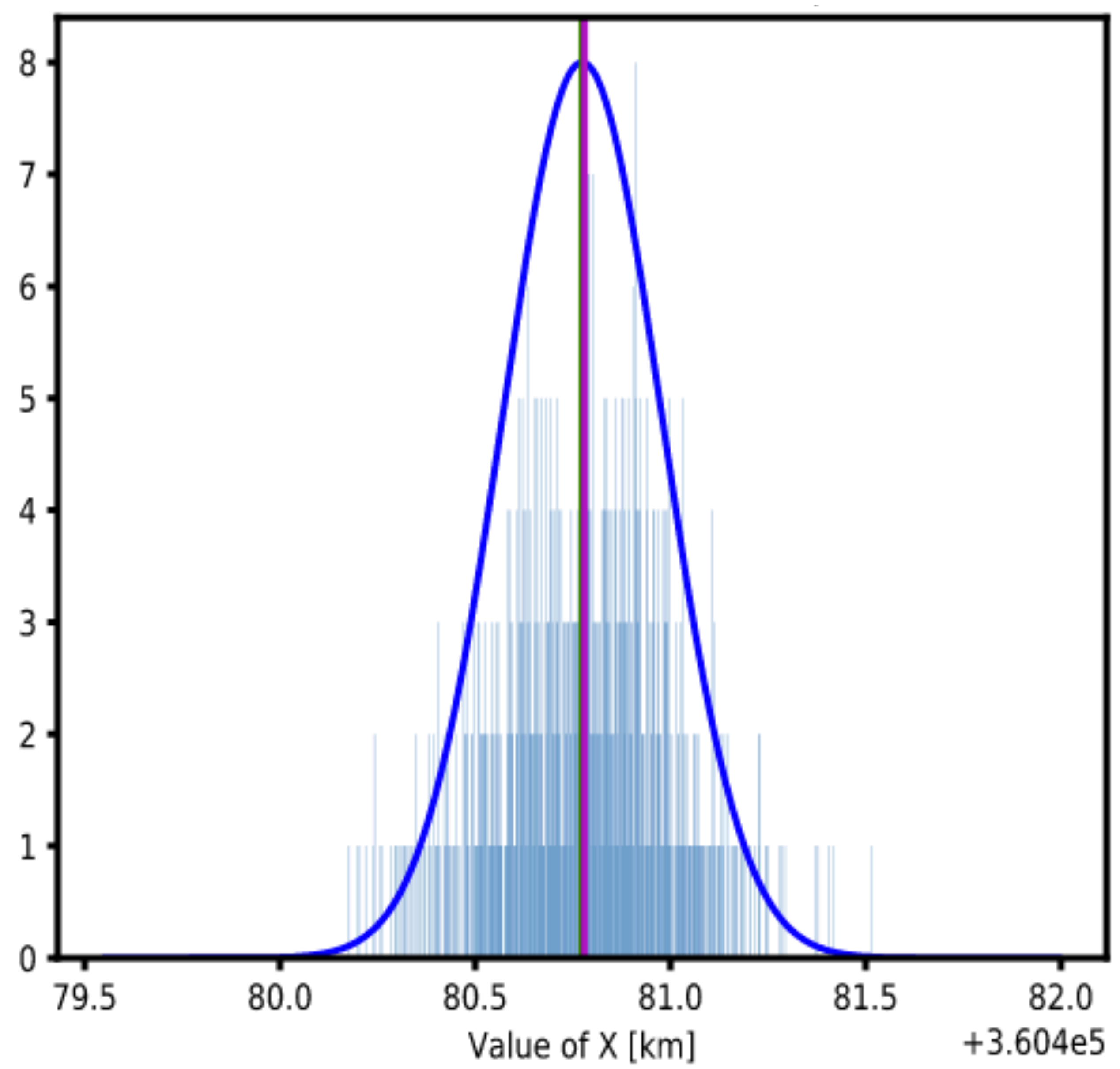
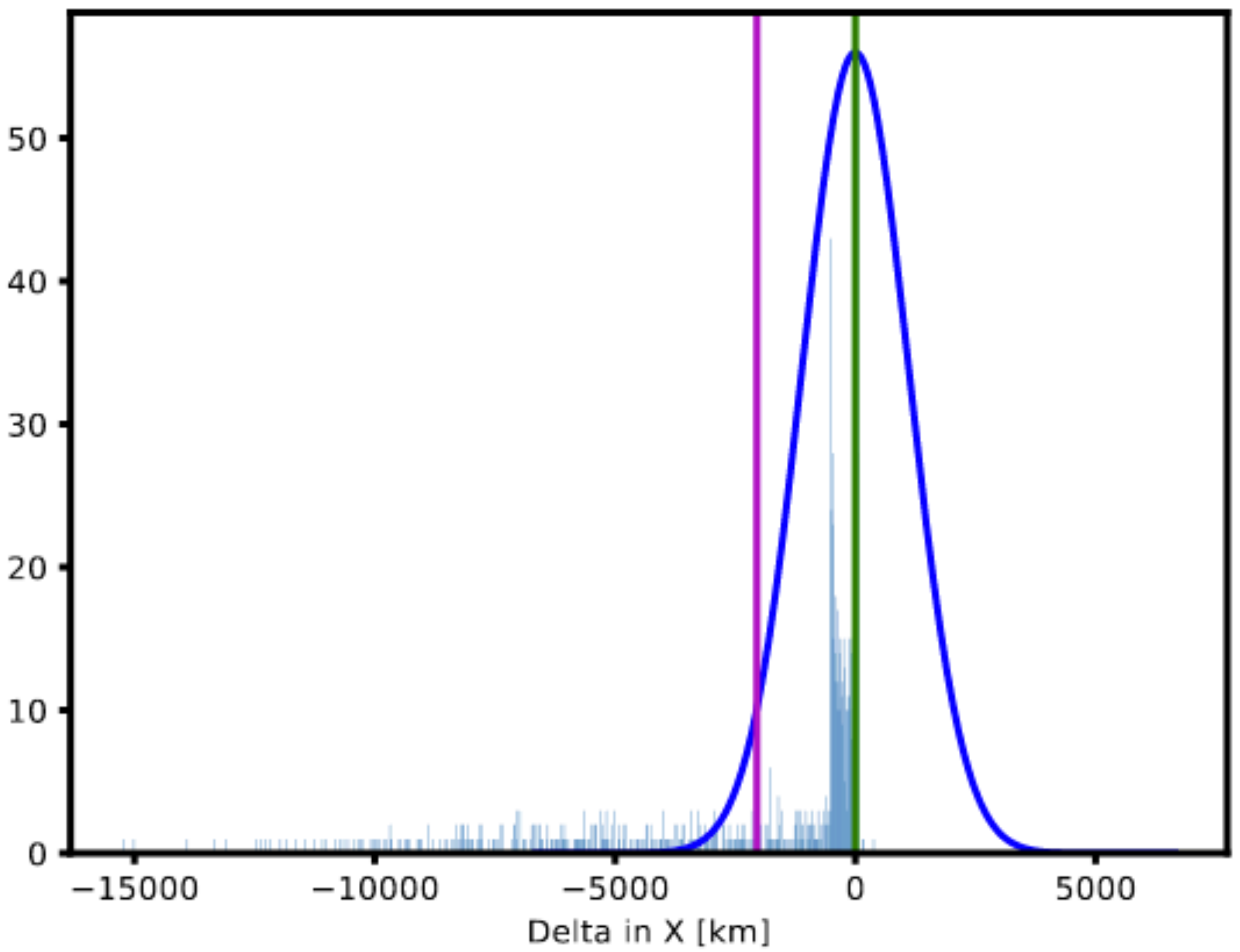
Proposed ΔV-free ballistic capture



Initial Gaussian uncertainty at leveraging maneuver

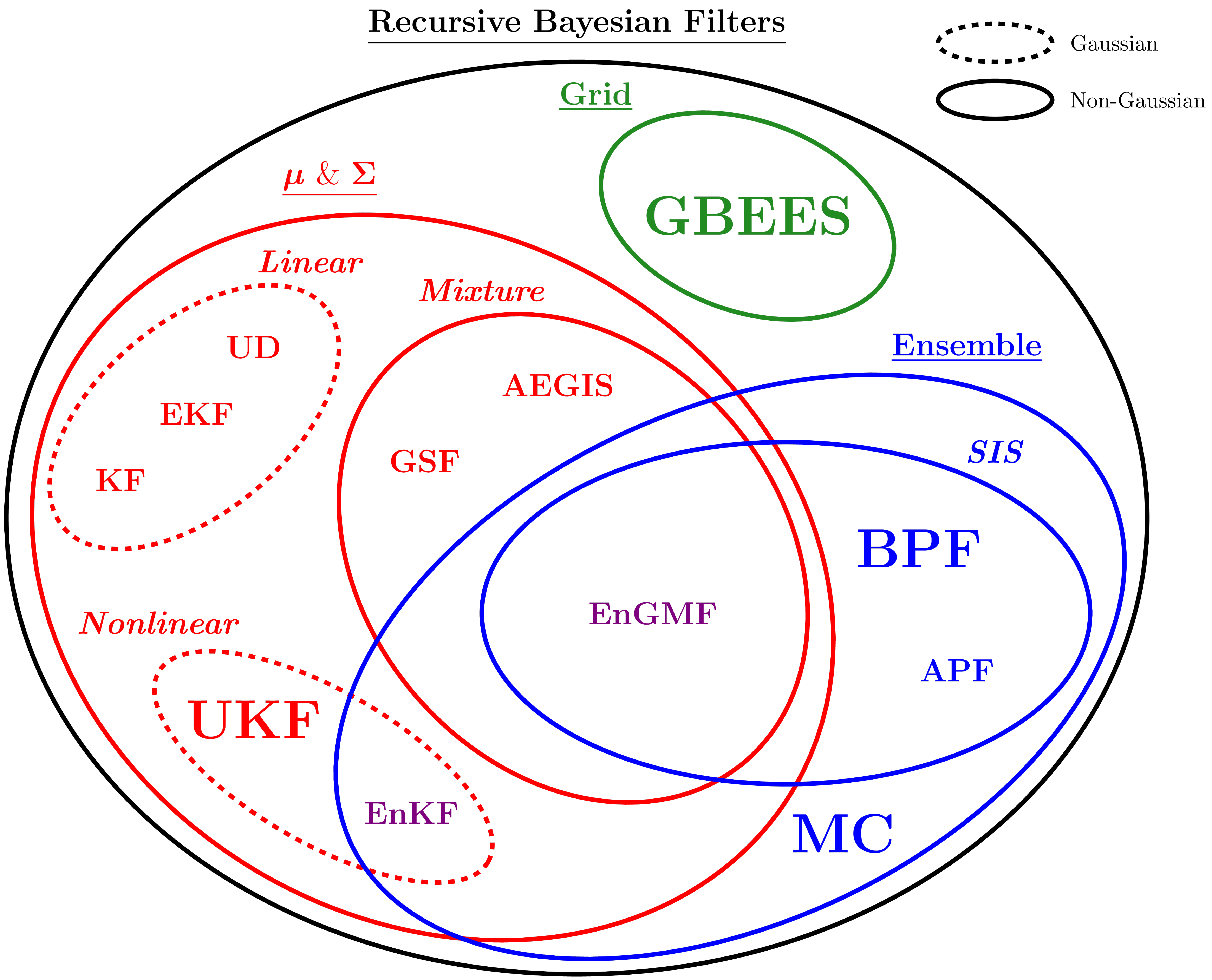


Final non-Gaussian uncertainty at Europa arrival

Key question: What are the temporal limits of Gaussian filters in the Jovian regime (or elsewhere), and when might it be necessary to implement non-Gaussian filters?

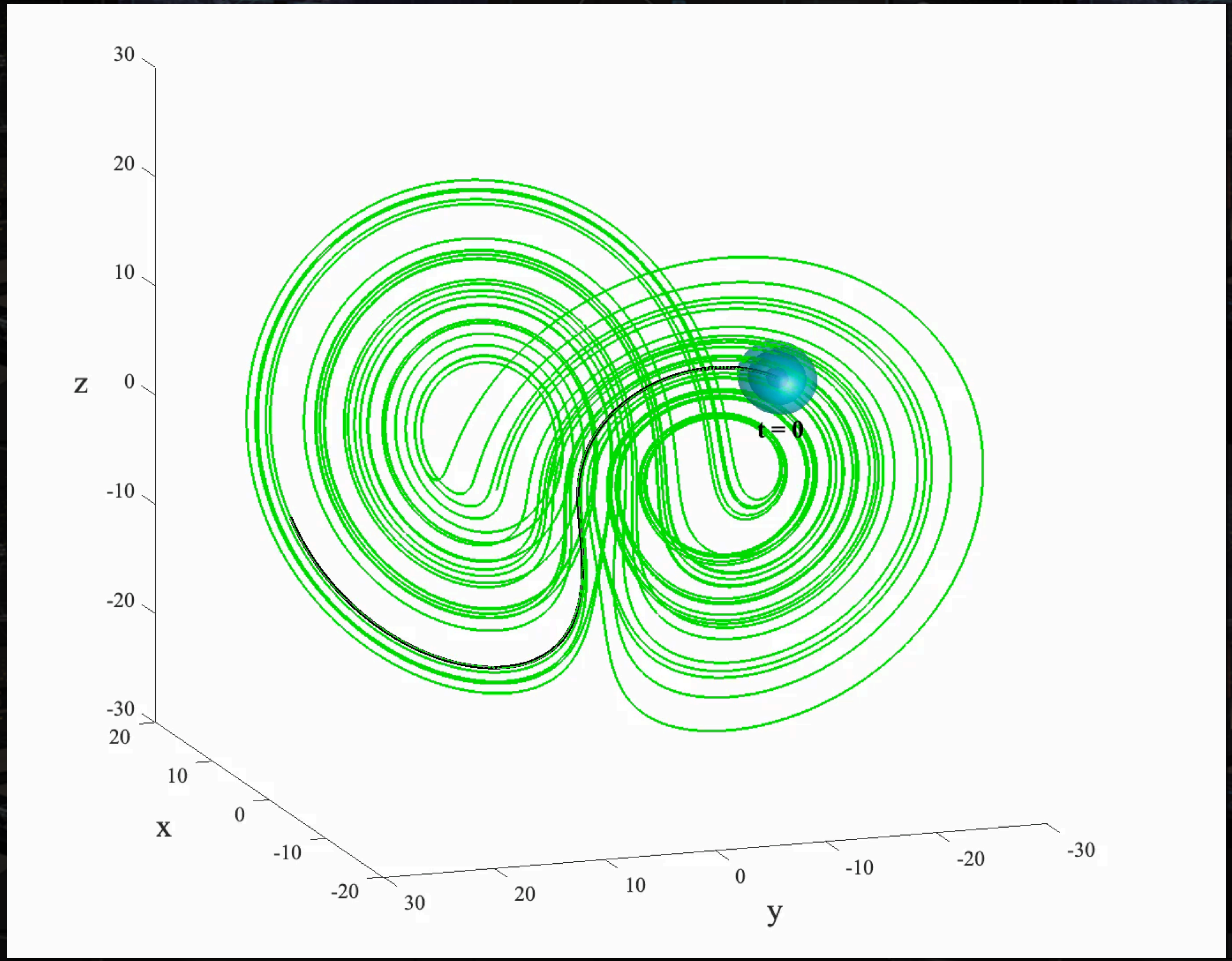
McElrath, Tim, et al. "Navigating low-energy trajectories to land on the surface of Europa." NTRS. (2021)

Current Landscape of Recursive Bayesian Filters



Motivation for New Non-Gaussian Filter

- To address the shortcomings of Gaussian filters, we utilize...



Grid-based Bayesian Estimation Exploiting Sparsity (GBEES)

An efficient Bayesian estimation method for representing and propagating uncertainty

- GBEES is a **2nd-order accurate**, Godunov finite volume method that **treats probability as a fluid**, flowing the PDF through phase space subject to the dynamics of the system
- Can handle deterministic/stochastic systems while **maintaining resolution**

Grid-based Bayesian Estimation Exploiting Sparsity (GBEES)

- GBEES consists of two distinct processes, one performed in **continuous-time**, the other in **discrete-time**:

1. The probability distribution function $p_{\mathbf{x}}(\mathbf{x}', t)$ is continuous-time marched via the **Fokker-Planck Equation**:

$$\frac{\partial p_{\boldsymbol{x}}(\boldsymbol{x}', t)}{\partial t} = -\frac{\partial f_i(\boldsymbol{x}', t) p_{\boldsymbol{x}}(\boldsymbol{x}', t)}{\partial x_i'} + \frac{1}{2}\frac{\partial^2 q_{ij} p_{\boldsymbol{x}}(\boldsymbol{x}', t)}{\partial x_i' \partial x_j'}$$

   - f_i: advection (EOMs) in the i^{th} dimension
   - q_{ij}: $(i, j)^{\text{th}}$ element of the spectral density ($Q = 0$, PDE is hyperbolic)

2. At discrete-time interval t_k, measurement $\boldsymbol{y}_k$ updates $p_{\boldsymbol{x}}(\boldsymbol{x}', t)$ via **Bayes' Theorem**:

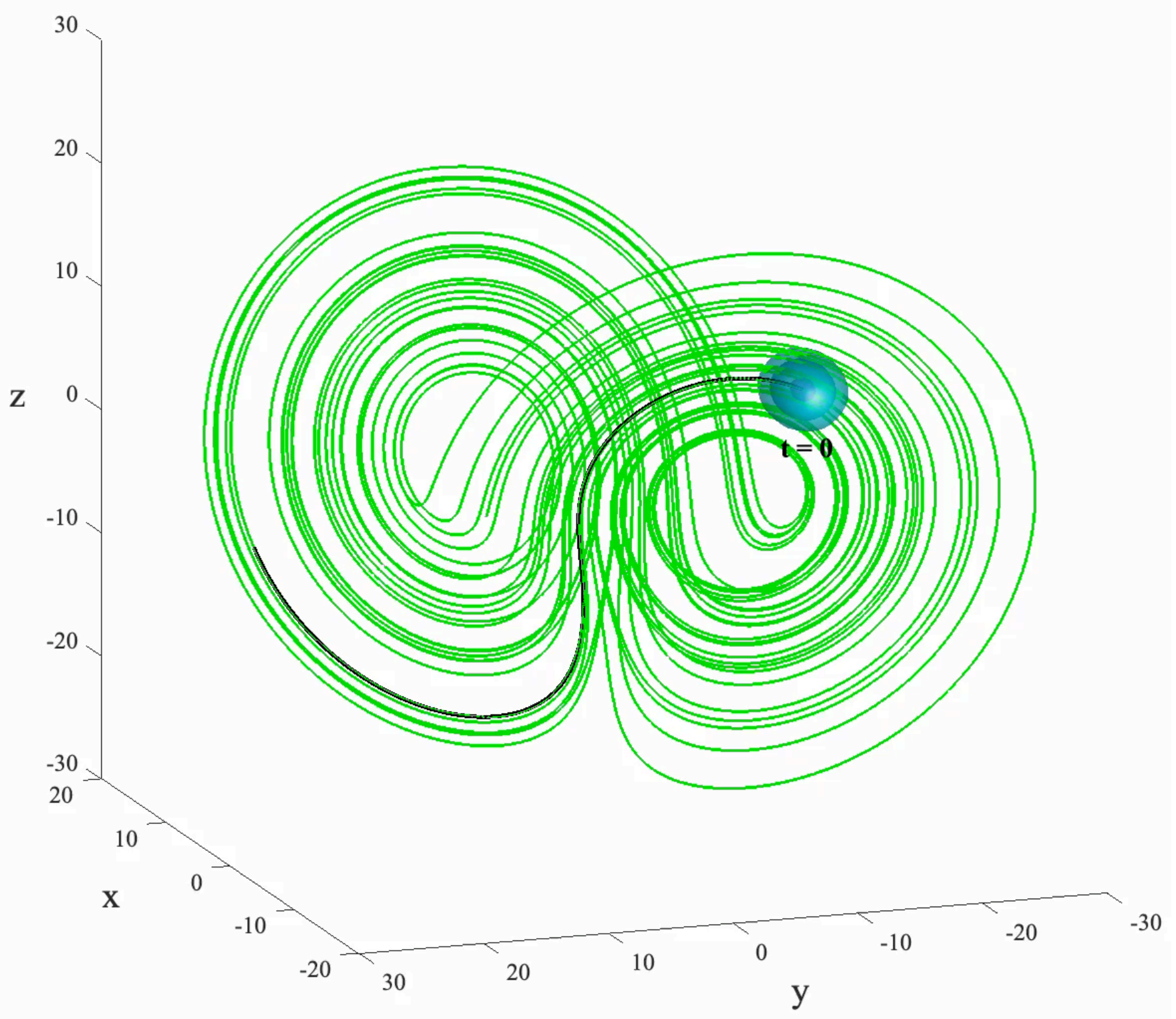
$$p_{\boldsymbol{x}}(\boldsymbol{x}', t_{k+}) = \frac{p_{\boldsymbol{y}}(\boldsymbol{y}_k \,|\, \boldsymbol{x}') p_{\boldsymbol{x}}(\boldsymbol{x}', t_{k-})}{C}$$

   - $p_{\mathbf{x}}(\mathbf{x}', t_{k+})$: a posteriori distribution
   - $p_{\mathbf{y}}(\mathbf{y}_k \,|\, \mathbf{x}')$: measurement distribution
   - $p_{\mathbf{x}}(\mathbf{x}', t_{k-})$: a priori distribution
   - C: normalization constant

Grid-based Bayesian Estimation Exploiting Sparsity (GBEES)

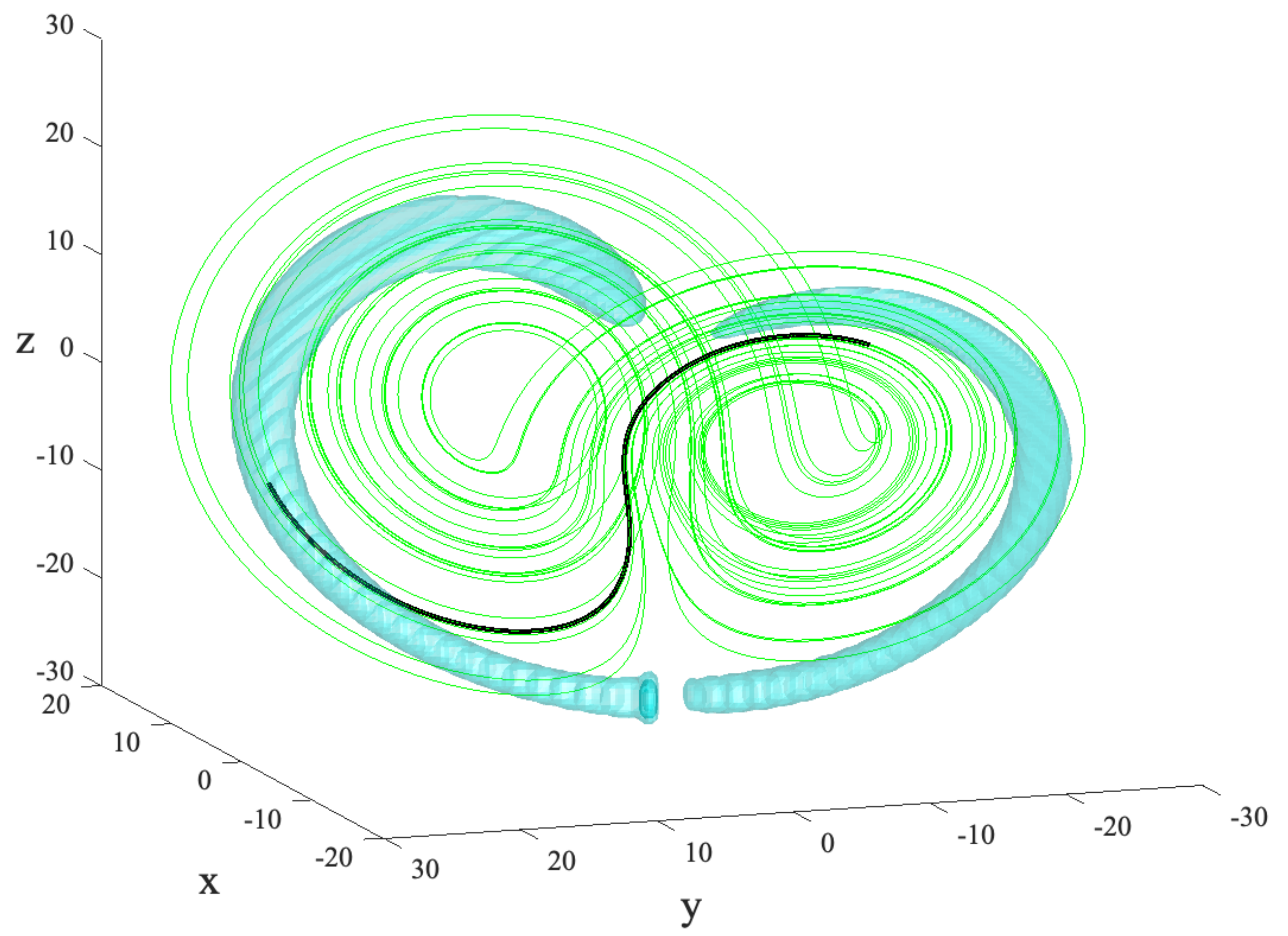
Continuous-time

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \frac{d\boldsymbol{x}}{dt} = \begin{bmatrix} \sigma(y-x) \\ -y-xz \\ -b(z+r)-xy \end{bmatrix}$$



Discrete-time

$$P(A\,|\,B) = \frac{P(B\,|\,A)\,P(A)}{P(B)}$$

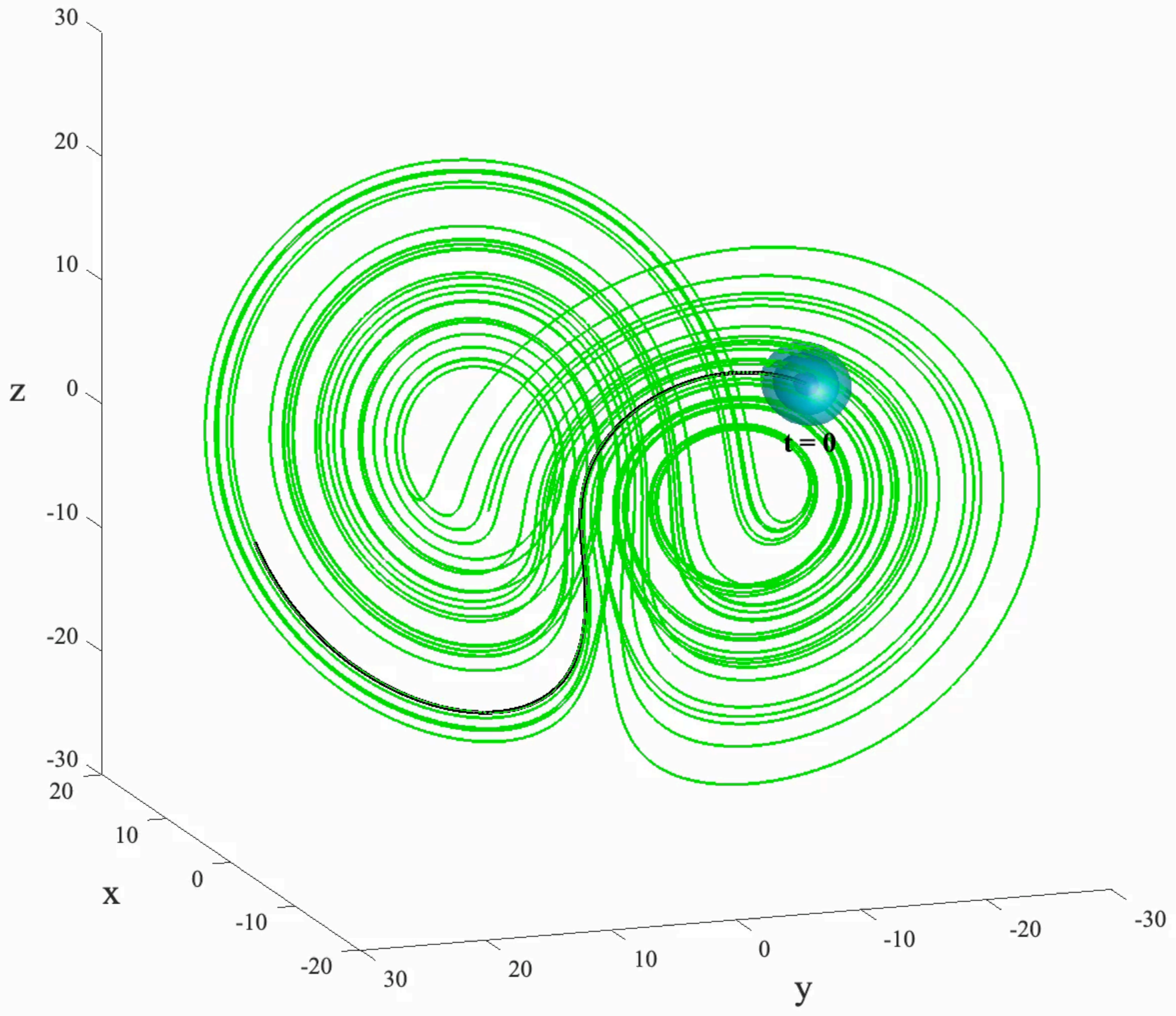


a priori

Bewley, Thomas, et al. "Efficient grid-based bayesian estimation of nonlinear low-dimensional systems with sparse non-gaussian pdfs." Automatica. (2012)

Continuous-time

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \frac{d\boldsymbol{x}}{dt} = \begin{bmatrix} \sigma(y-x) \\ -y-xz \\ -b(z+r)-xy \end{bmatrix}$$



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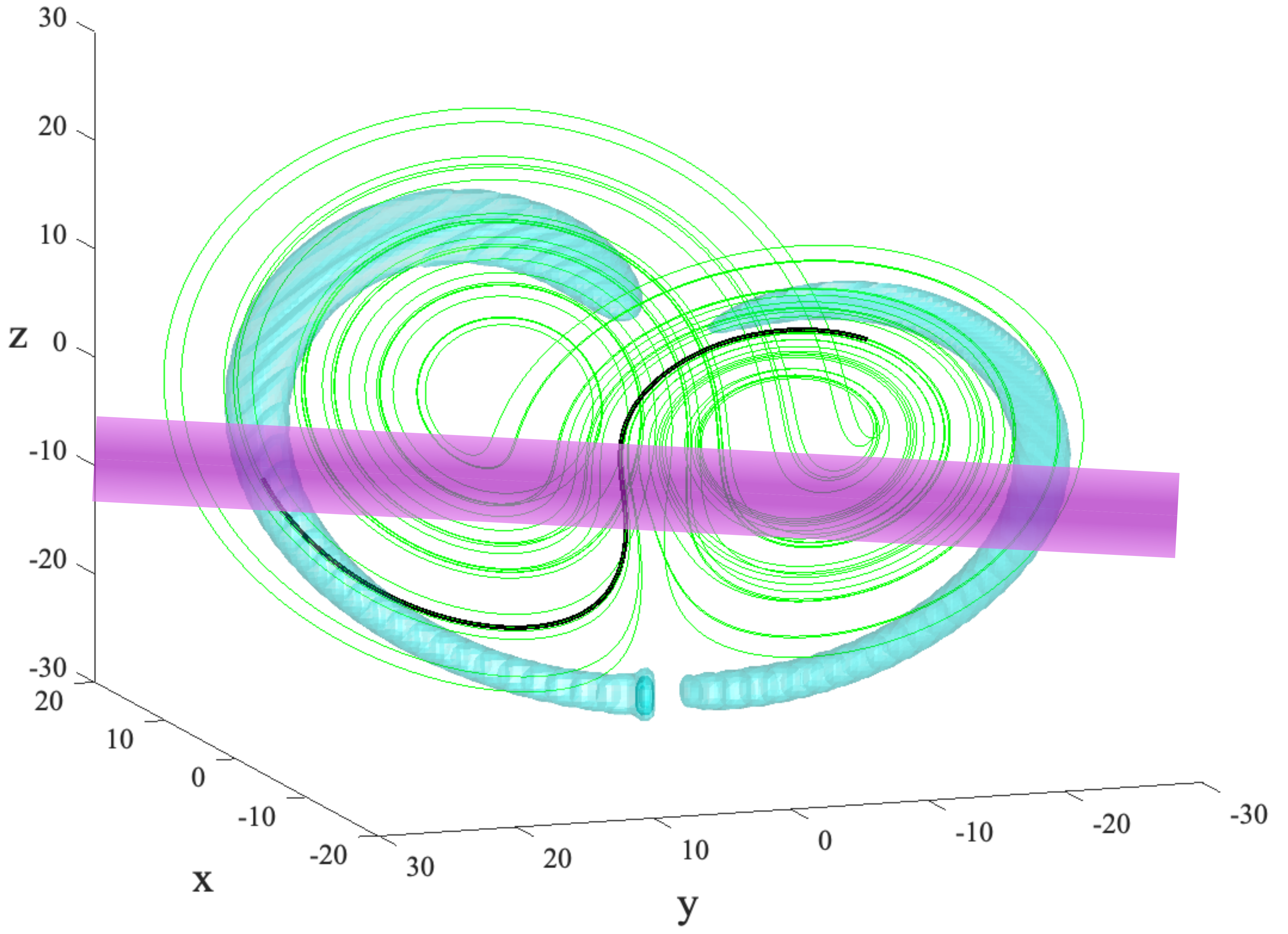
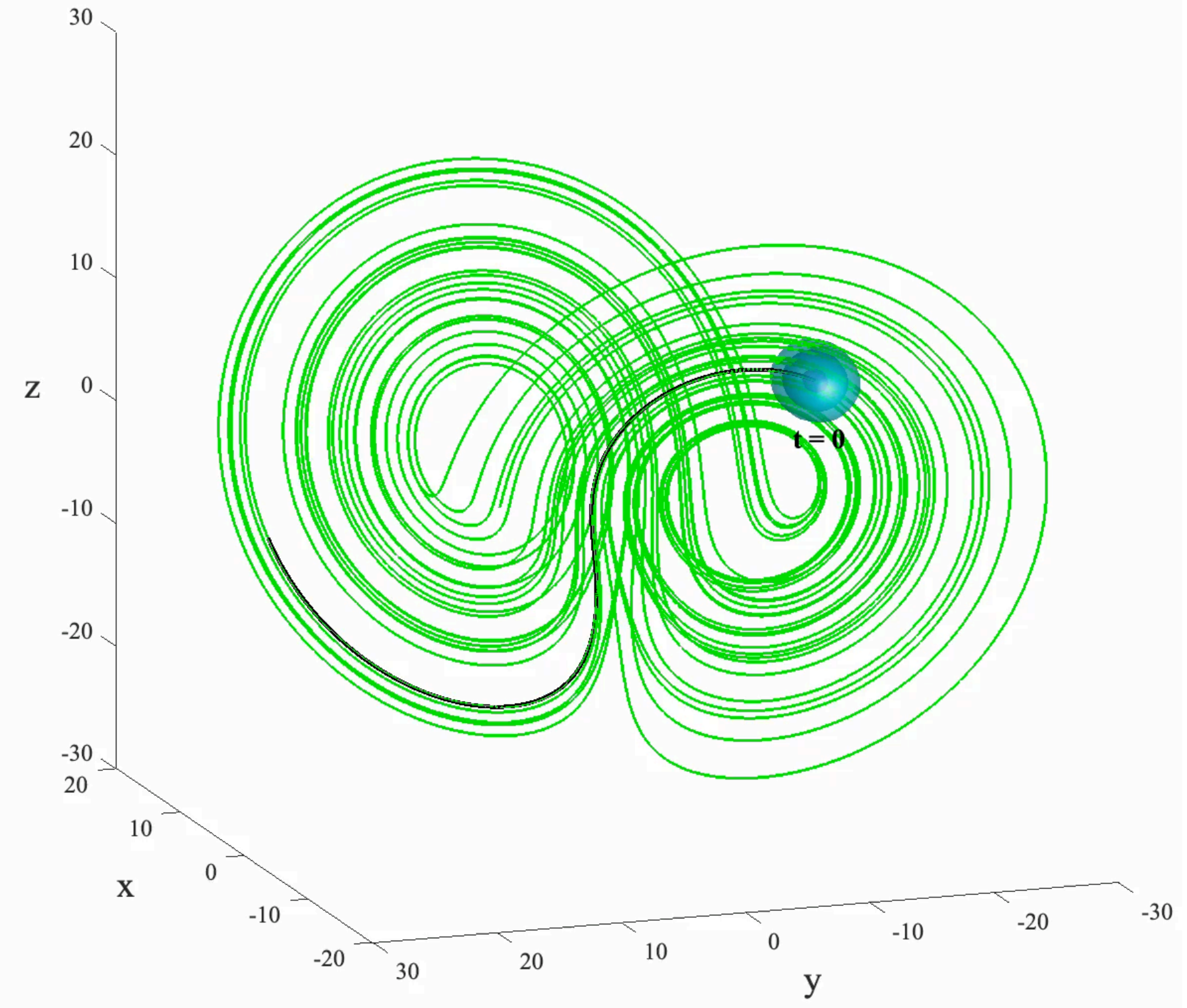


a priori × **likelihood**

Bewley, Thomas, et al. "Efficient grid-based bayesian estimation of nonlinear low-dimensional systems with sparse non-gaussian pdfs." Automatica. (2012)

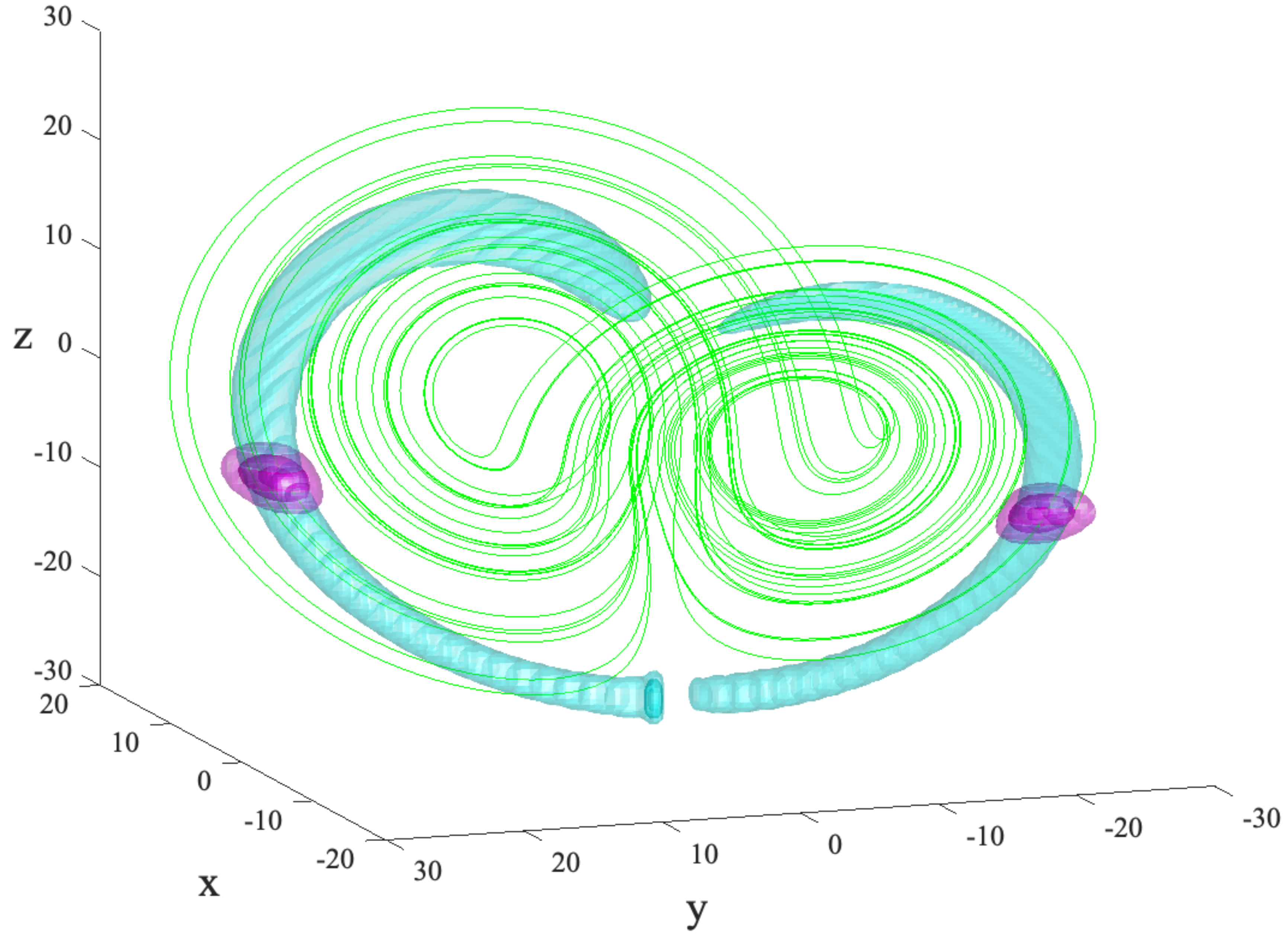
Continuous-time

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \frac{d\boldsymbol{x}}{dt} = \begin{bmatrix} \sigma(y-x) \\ -y - xz \\ -b(z+r) - xy \end{bmatrix}$$



Discrete-time

$$P(A\,|\,B) = \frac{P(B\,|\,A)\,P(A)}{P(B)}$$

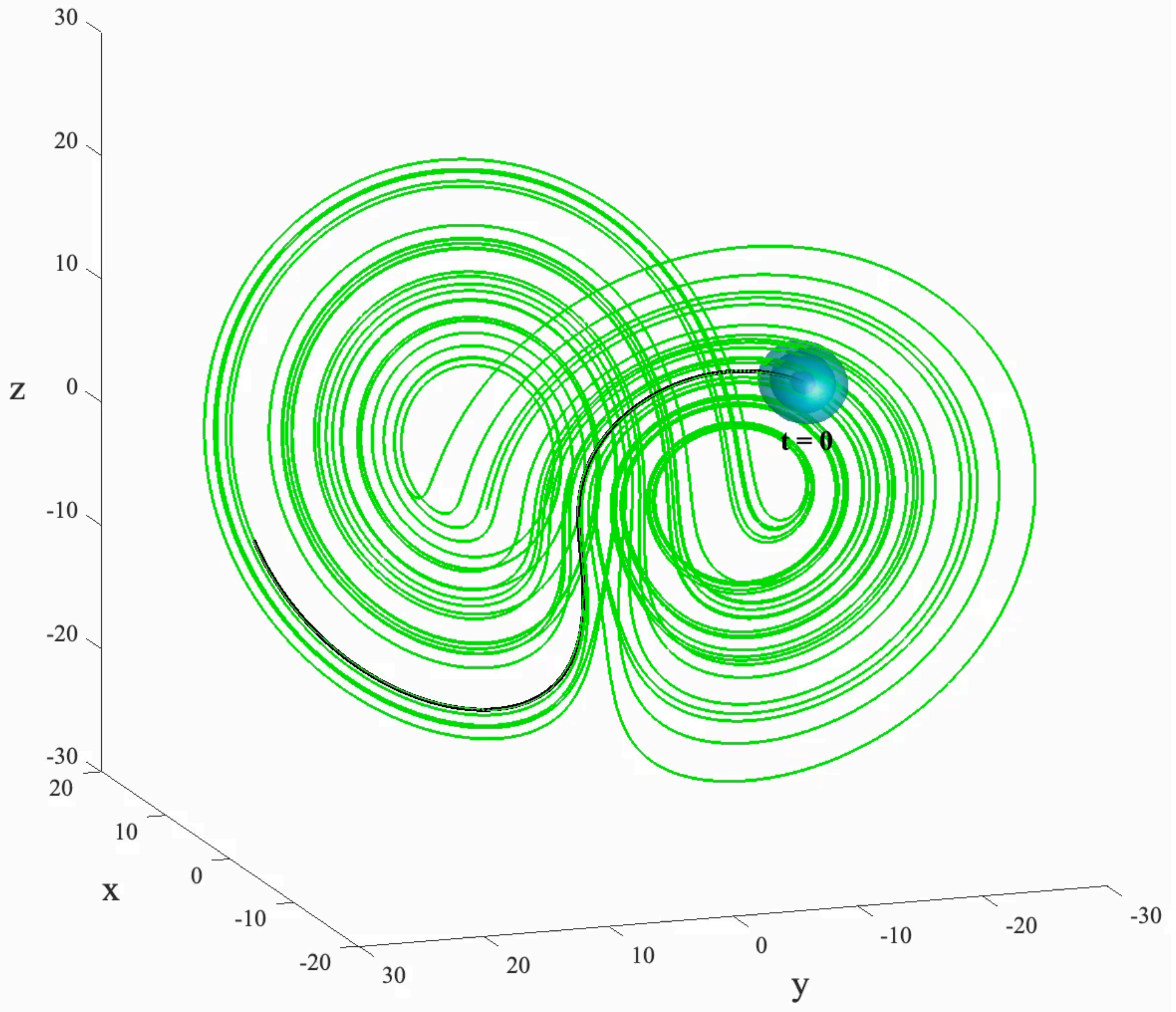


a priori × **likelihood** = ***a posteriori***

Bewley, Thomas, et al. "Efficient grid-based bayesian estimation of nonlinear low-dimensional systems with sparse non-gaussian pdfs." Automatica. (2012)

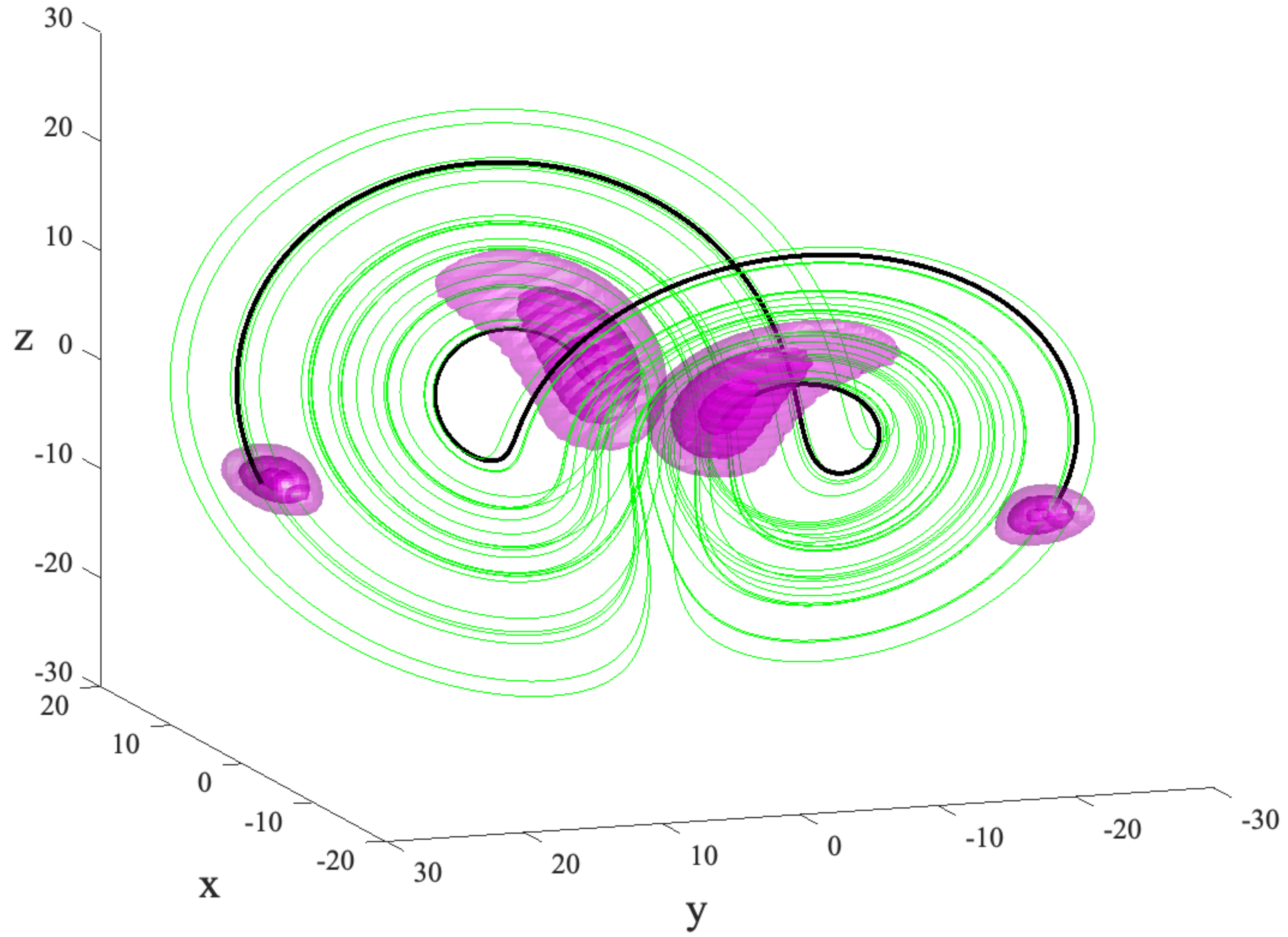
Continuous-time

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \frac{d\boldsymbol{x}}{dt} = \begin{bmatrix} \sigma(y-x) \\ -y-xz \\ -b(z+r)-xy \end{bmatrix}$$



Discrete-time

$$P(A\,|\,B) = \frac{P(B\,|\,A)\,P(A)}{P(B)}$$



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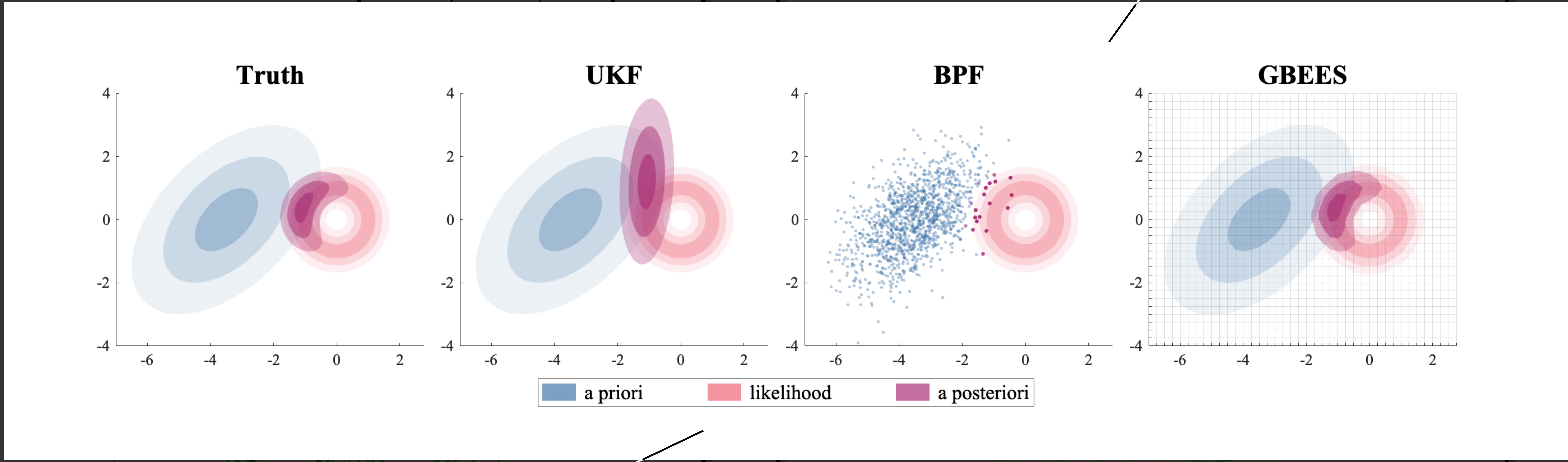
Bewley, Thomas, et al. "Efficient grid-based bayesian estimation of nonlinear low-dimensional systems with sparse non-gaussian pdfs." Automatica. (2012)

Grid-based **Bayesian Estimation** Exploiting Sparsity (GBEES)

$n_{\text{particles}} = n_{\text{grid cells}}$

Continuous-time

Discrete-time



$$x = \begin{bmatrix} x \\ y \end{bmatrix}, \quad h(x) = |x|$$

a priori × **likelihood** = ***a posteriori***

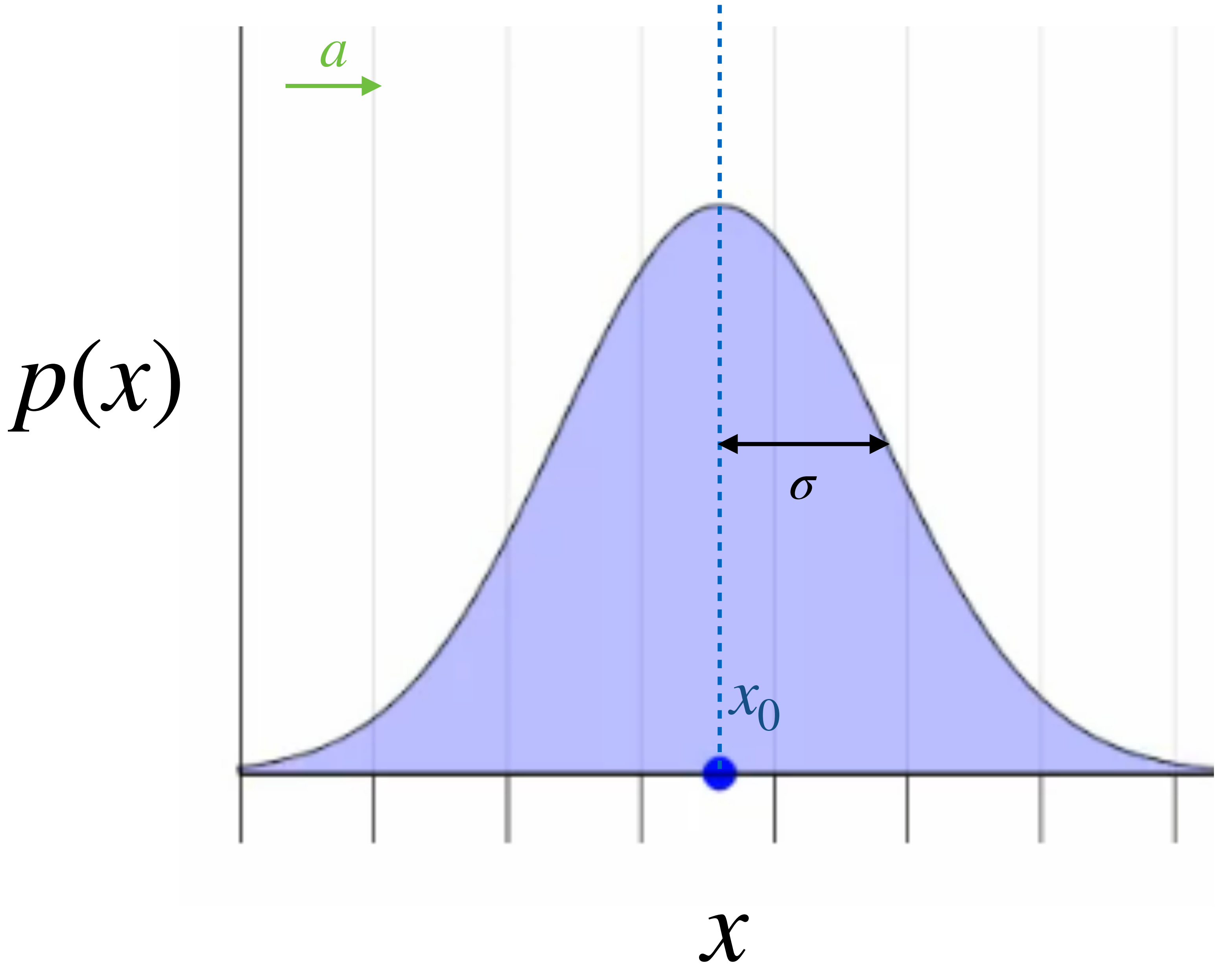
Bewley, Thomas, et al. "Efficient grid-based bayesian estimation of nonlinear low-dimensional systems with sparse non-gaussian pdfs." Automatica. (2012)

Grid-based Bayesian Estimation Exploiting Sparsity (GBEES)

- Consider a 1-dimensional, linear test example:

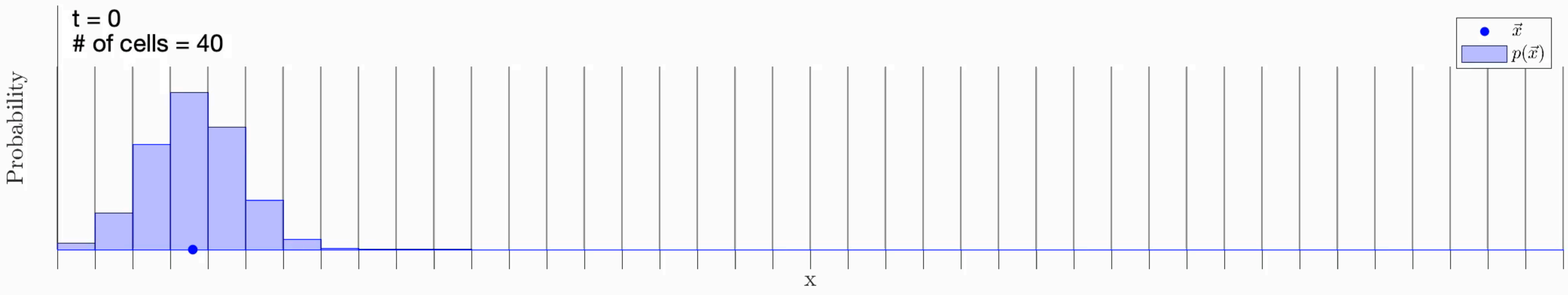
$$\boldsymbol{x} = [x], \quad \frac{d\boldsymbol{x}}{dt} = [a], \quad a > 0$$

- Initial observation of $x(t)$ results in a Gaussian PDF $p(x)$ centered about x_0 with standard deviation σ

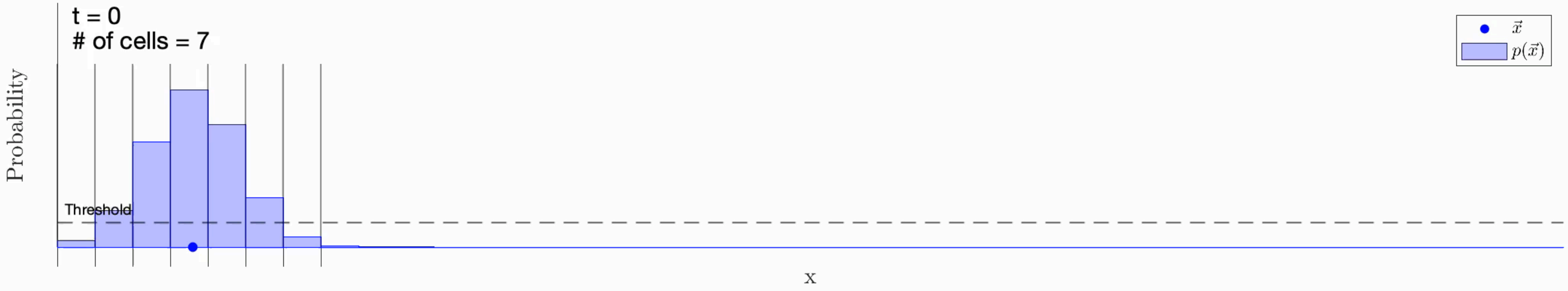


How does $p(x)$, governed by $d\boldsymbol{x}/dt$, change with respect to t?

Ignoring sparsity



Exploiting sparsity

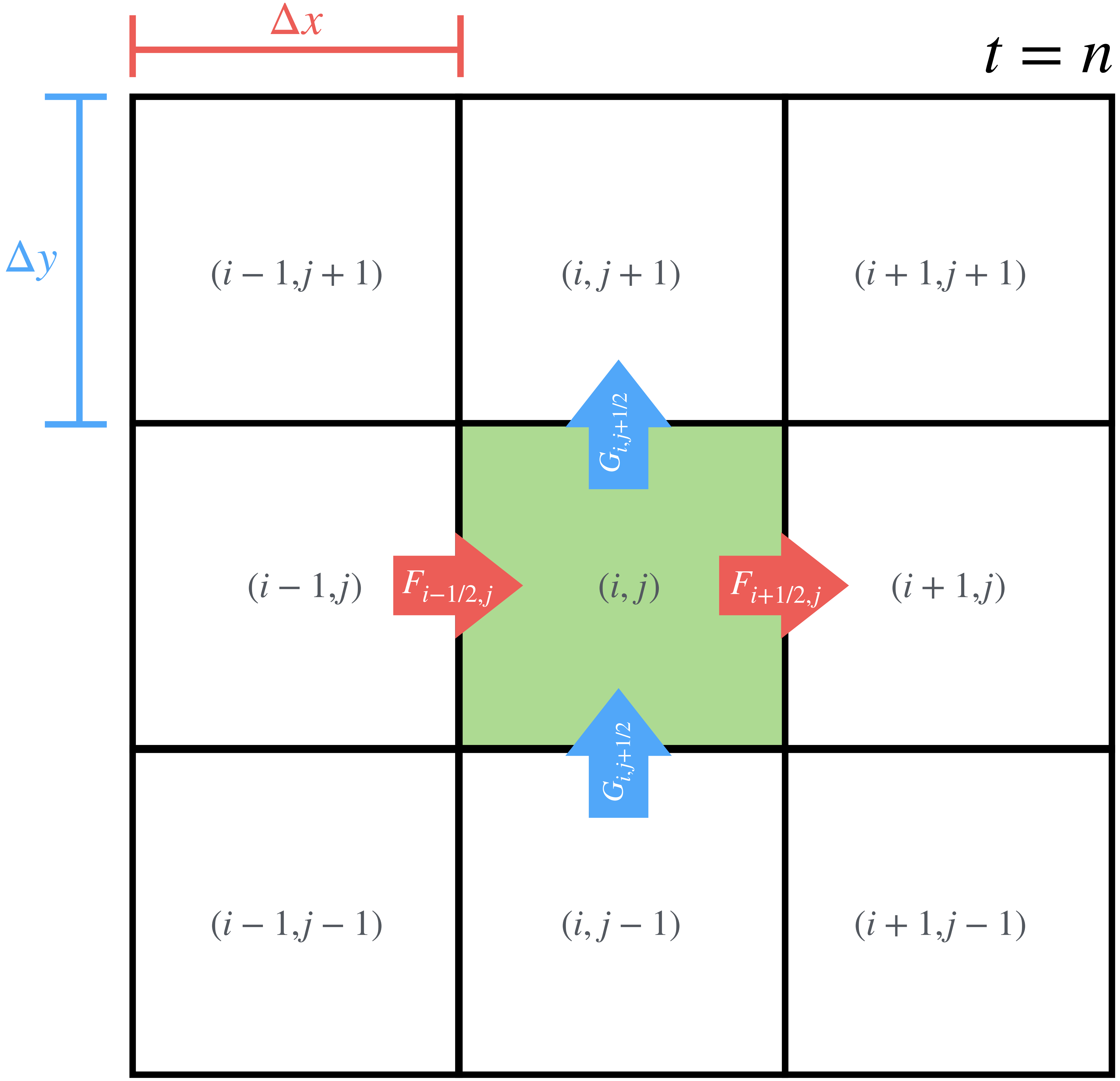


Not GBEES, just a visual aid

General Formulation

Godunov Upwind Scheme - Fully Discretized, 2nd-order, Taylor Approximation

- A Godunov-type finite volume method implemented on a uniform Cartesian 2D mesh



$$\frac{p_{i,j}^{n+1} - p_{i,j}^{n}}{\Delta t} = -\frac{F_{i+1/2,j}^{n} - F_{i-1/2,j}^{n}}{\Delta x} - \frac{G_{i,j+1/2}^{n} - G_{i,j-1/2}^{n}}{\Delta y}$$

- $p_{i,j}^{n+1}$: probability at time step $n+1$ at cell (i, j)
- $p_{i,j}^{n}$: probability at time step n at cell (i, j)
- Δt: size of time step
- $F_{i-1/2,j}^{n}$: flux a half grid length back in the x-direction
- $F_{i+1/2,j}^{n}$: flux a half grid length forward in the x-direction
- $G_{i-1/2,j}^{n}$: flux a half grid length back in the y-direction
- $G_{i+1/2,j}^{n}$: flux a half grid length forward in the y-direction
- Δx: x-grid width
- Δy: y-grid width

Instead of flux being a function of volume and advection, flux is a function of probability and the equations of motion!

Goal: Find practical trajectories where orbital uncertainty becomes non-Gaussian and apply RBFs

Astrodynamic Applications: Three-Body Problem

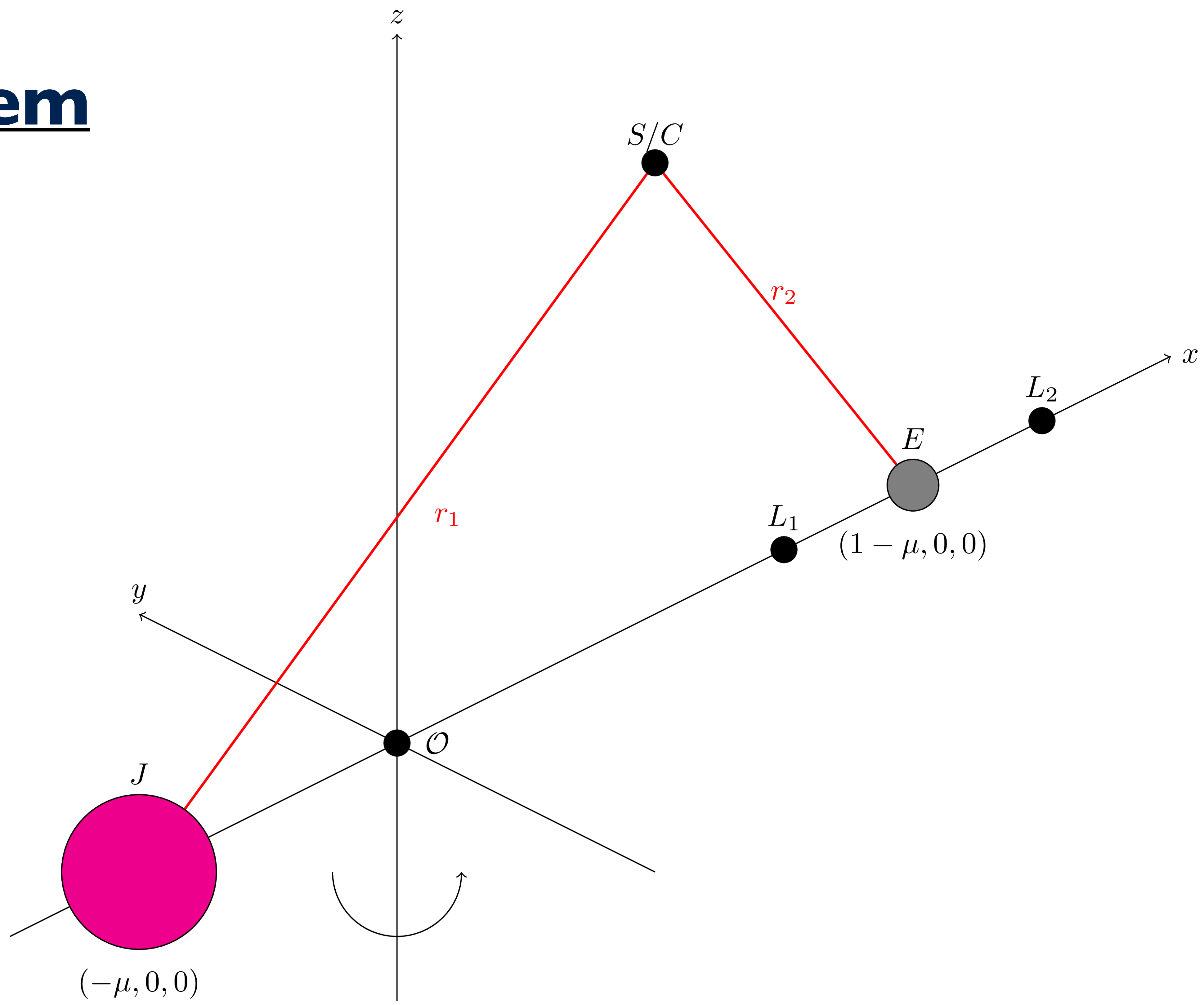
Circular Restricted Three-Body Problem (CR3BP)

- We look to apply the developed framework to orbital uncertainty propagation

Circular Restricted Three-Body Problem

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}, \quad \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ 2\dot{y} + \Omega_x \\ -2\dot{x} + \Omega_y \\ \Omega_z \end{bmatrix}$$

where $\Omega(x, y) = \dfrac{x^2 + y^2}{2} + \dfrac{1-\mu}{r_1} + \dfrac{\mu}{r_2} + \dfrac{\mu(1-\mu)}{2}$



- We use initial conditions generated from the JPL Three-Body Periodic Orbit Catalog

Park, Ryan. "Jet Propulsion Laboratory Three-Body Periodic Orbit Catalog." (2024)

Astrodynamic Applications: Three-Body Problem

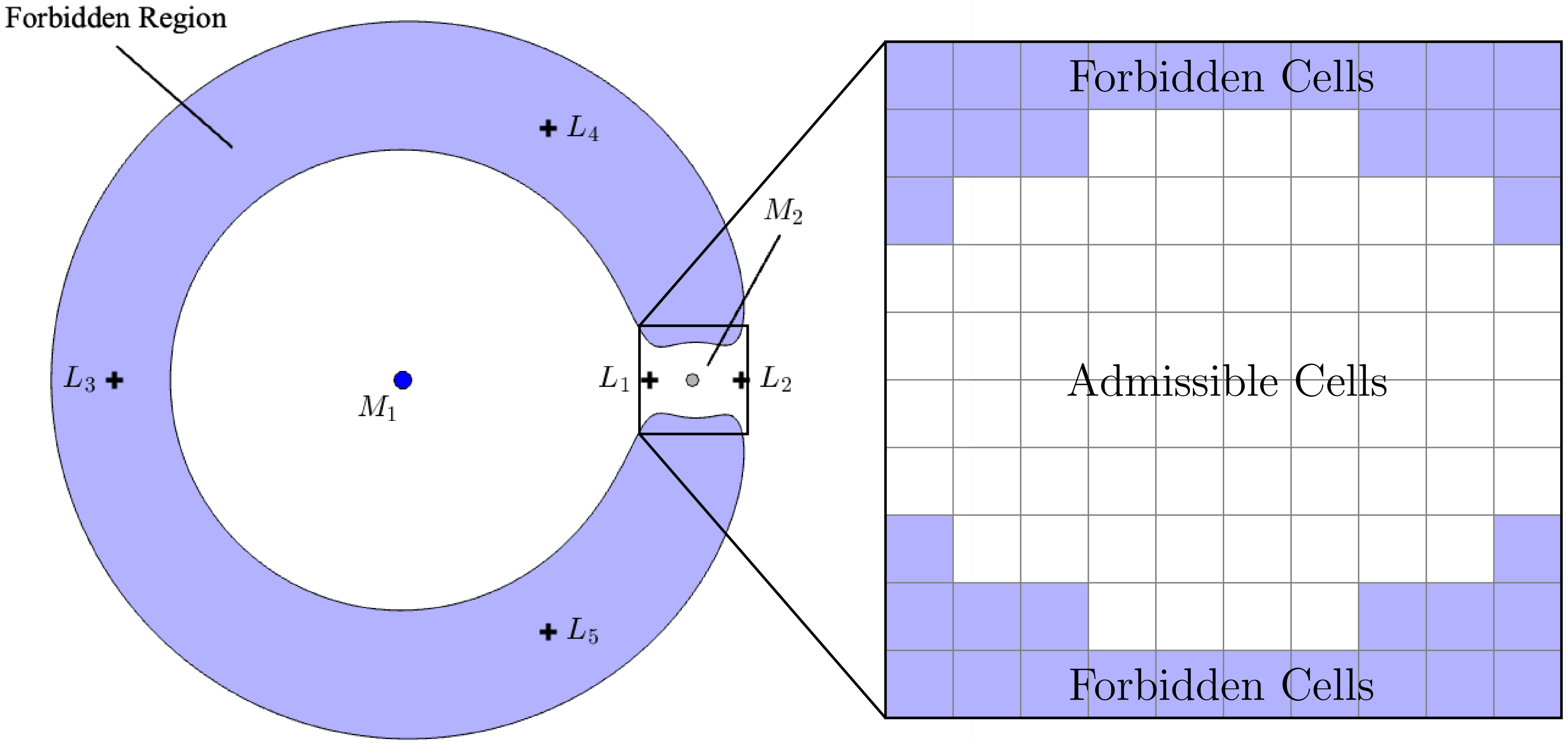
GBEES Jacobi Bounding

- One integral of motion exists for the CR3BP

$$C(x, y, z, \dot{x}, \dot{y}, \dot{z}) = x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} + \mu(1-\mu) - \dot{x}^2 - \dot{y}^2 - \dot{z}^2$$

- GBEES is a 2nd-order accurate numerical scheme, so C is not necessarily conserved
- Instead, we hardcode this requirement into the grid creation

  - ✳ When $\dot{x} = \dot{y} = \dot{z} = 0$, C is the zero-velocity curves
  - ✳ Phase space is discretized as would be done for GBEES



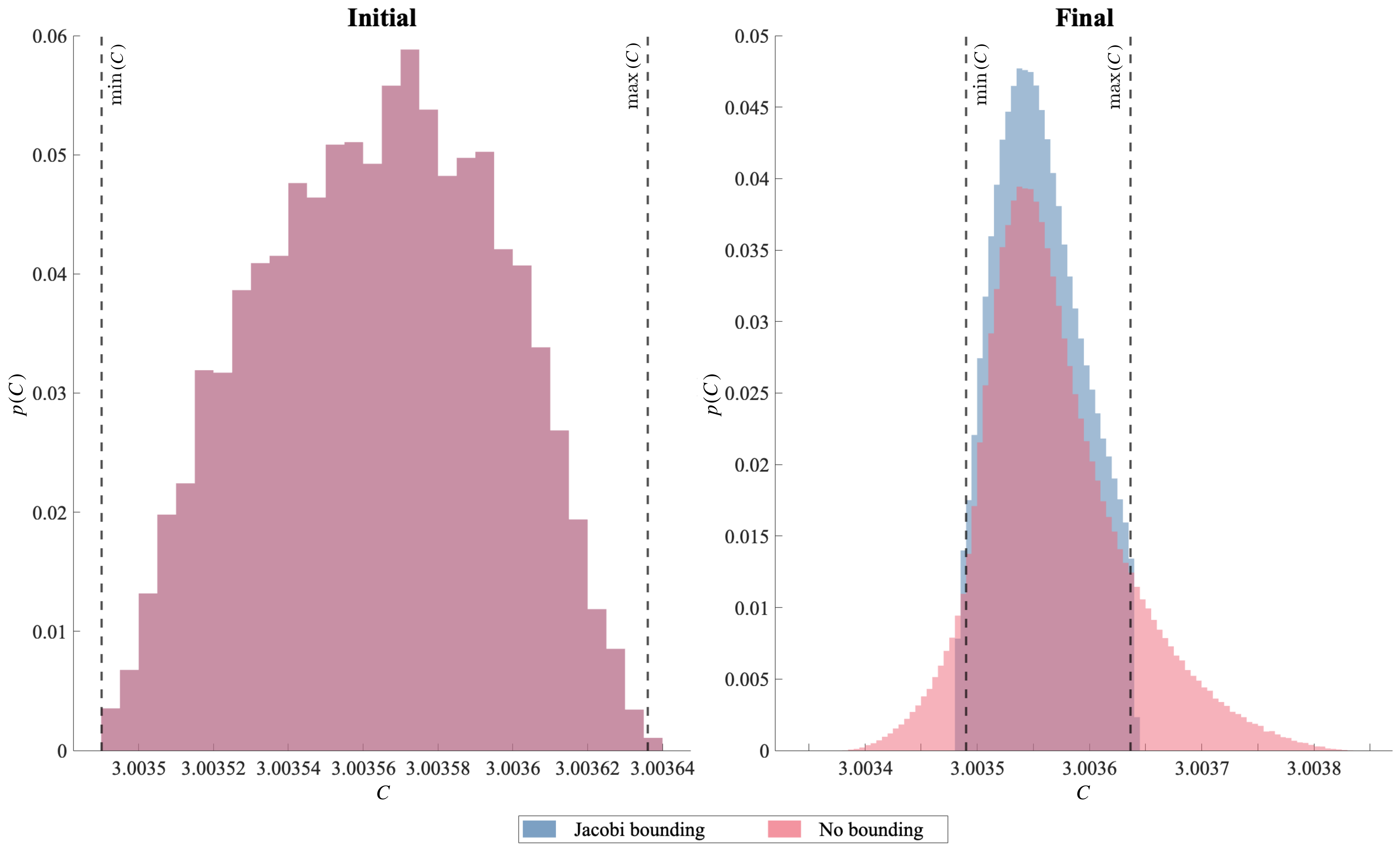
GBEES Jacobi Bounding

- One integral of motion exists for the CR3BP

$$C(x,y,z,\dot{x},\dot{y},\dot{z}) = x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} + \mu(1-\mu) - \dot{x}^2 - \dot{y}^2 - \dot{z}^2$$

- GBEES is 2nd-order accurate, so C is not necessarily conserved numerically
- Instead, we hardcode this requirement into the grid creation

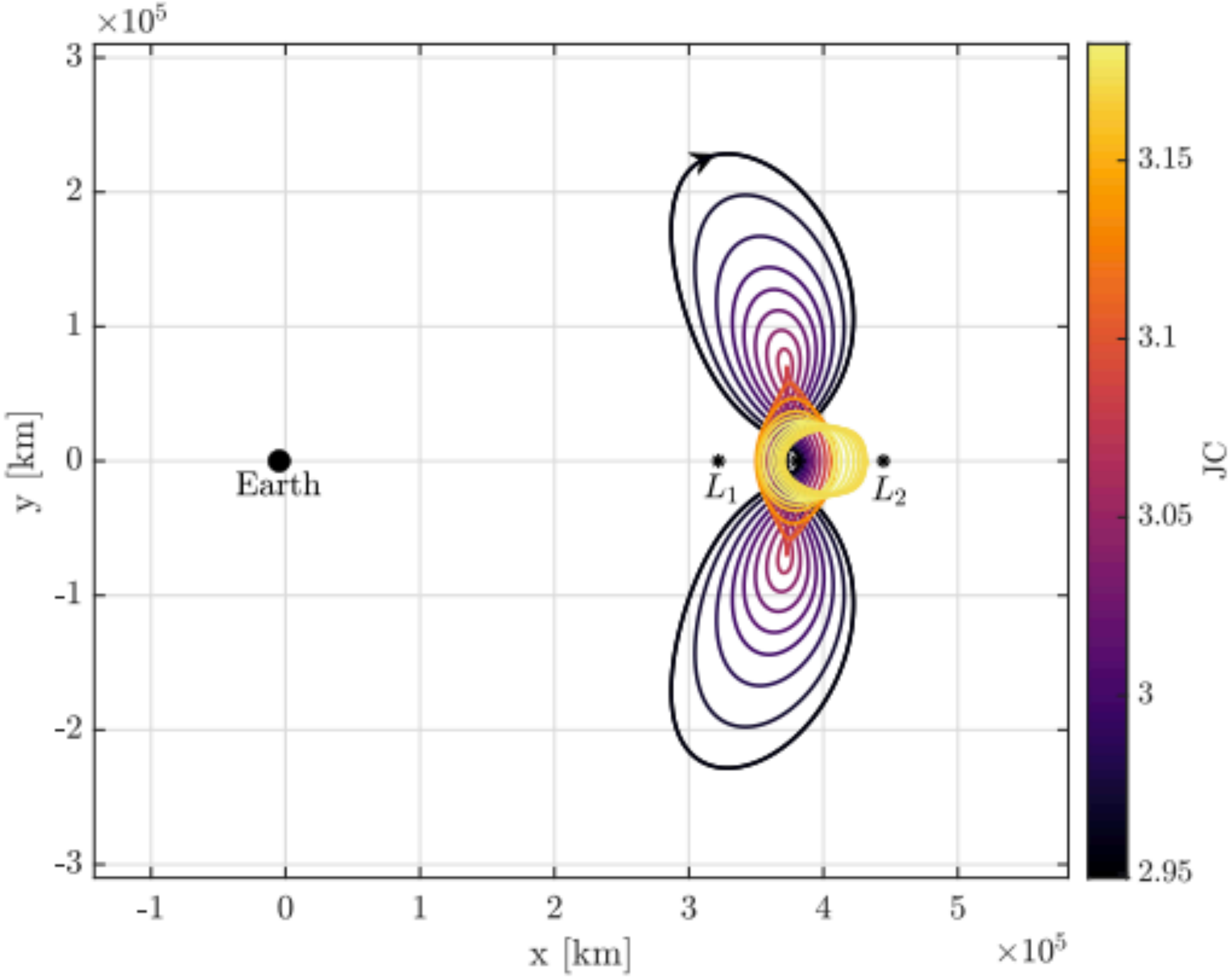
  * For the initial PDF, there exists a $\min(C)$ and a $\max(C)$
  * We ensure that all grid cells that are created in the propagation period are between these bounds

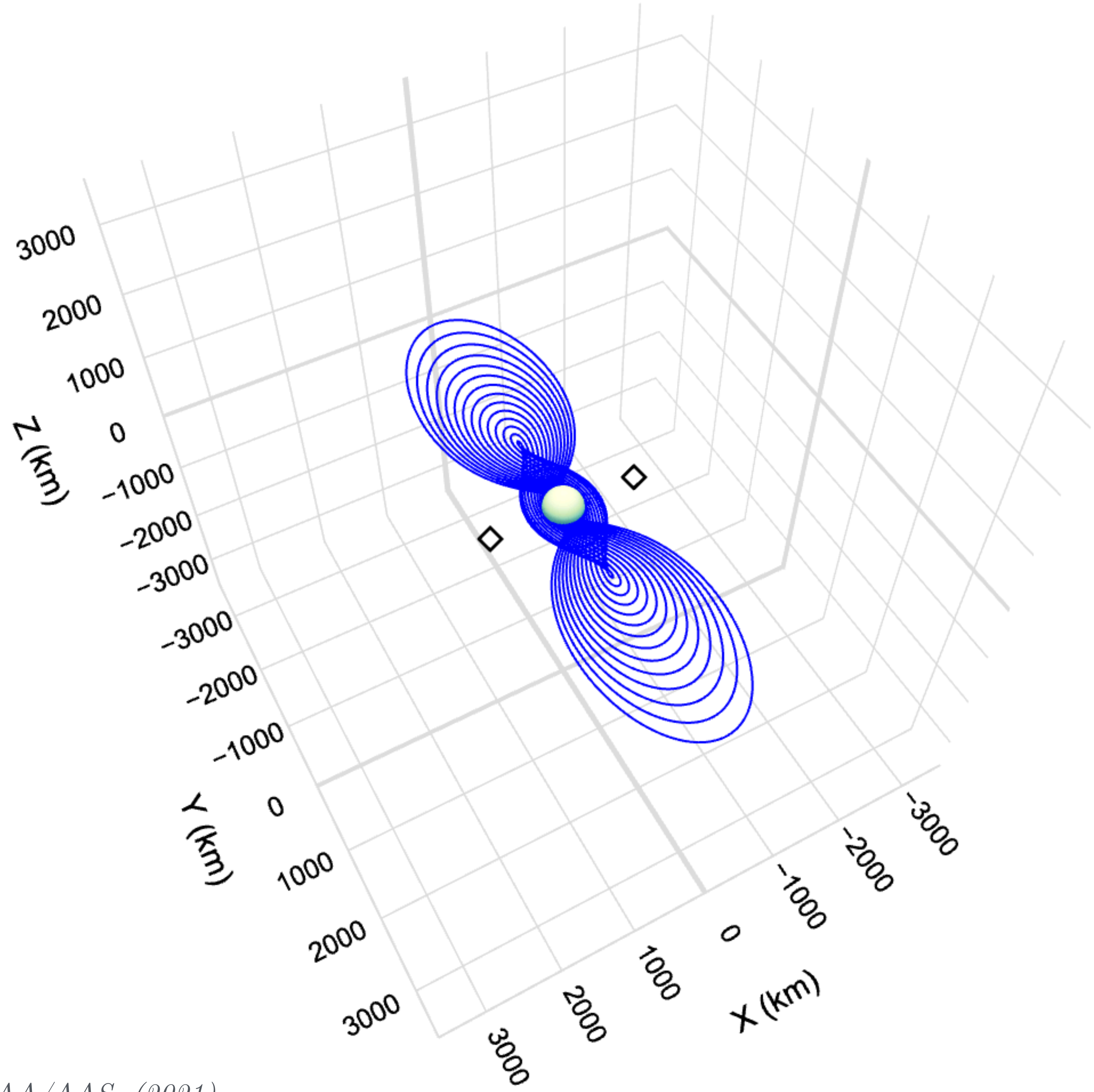


Circular Restricted Three-Body Problem (CR3BP)

Distant Prograde Orbits

- A family of planar, P_2-centered, stable/unstable periodic orbits that emerge from the dynamics of the CR3BP are Distant Prograde Orbits (DPOs)





Gupta, Maaninee et al. "Earth-Moon multi-body orbits to facilitate cislunar surveillance activities." AIAA/AAS. (2021)

Circular Restricted Three-Body Problem (CR3BP)

Distant Prograde Orbits

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<u>Stability is a function of C</u>

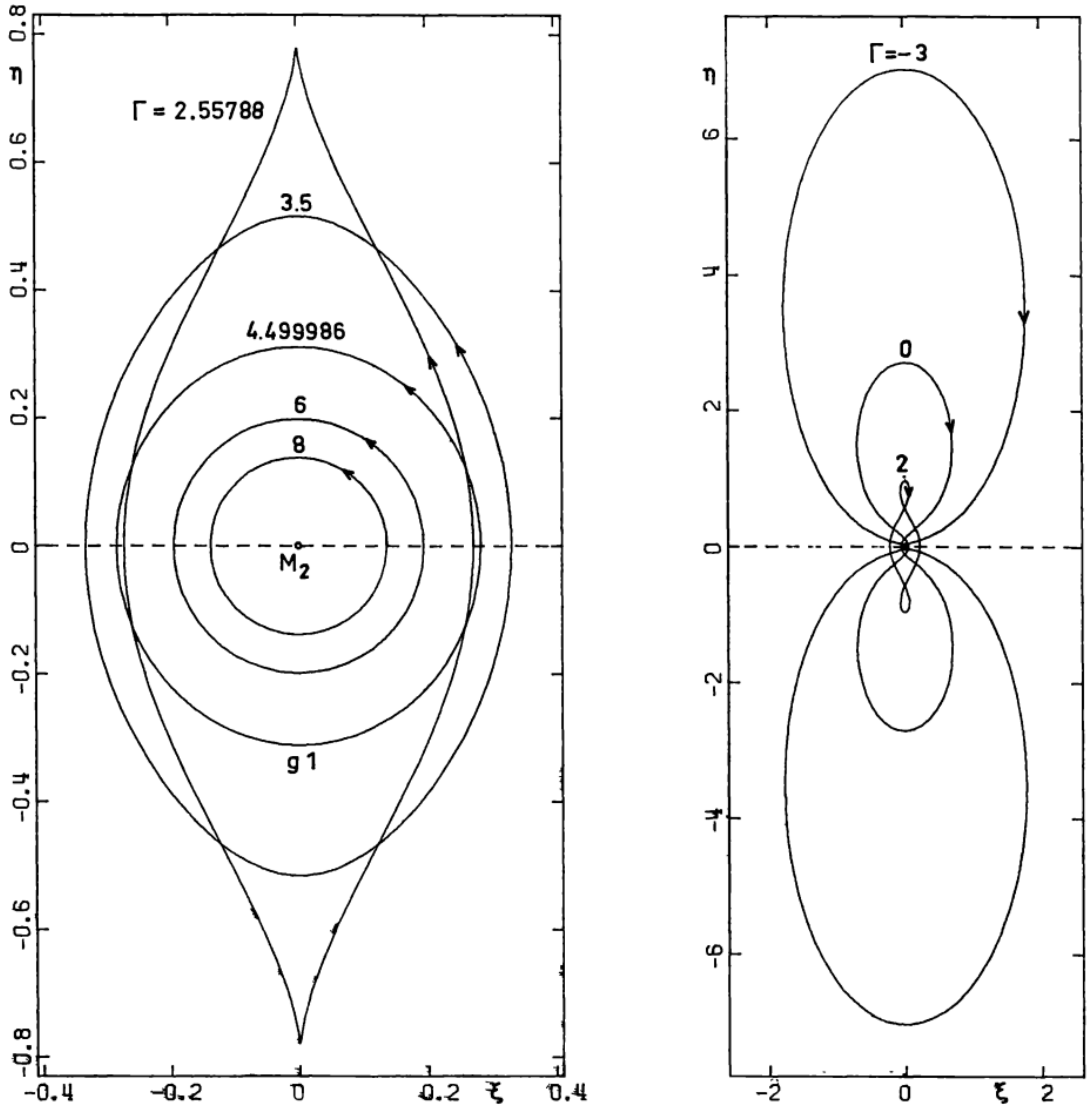

Fig. 4. Family g of periodic orbits. Note that the two parts of the figure have different scales. Orbits with $\Gamma > 4.499986$ are stable, the others are unstable

Henon, Michel. "Numerical exploration of the restricted problem, V." AAP. (1969)

Circular Restricted Three-Body Problem (CR3BP)

Distant Prograde Orbits

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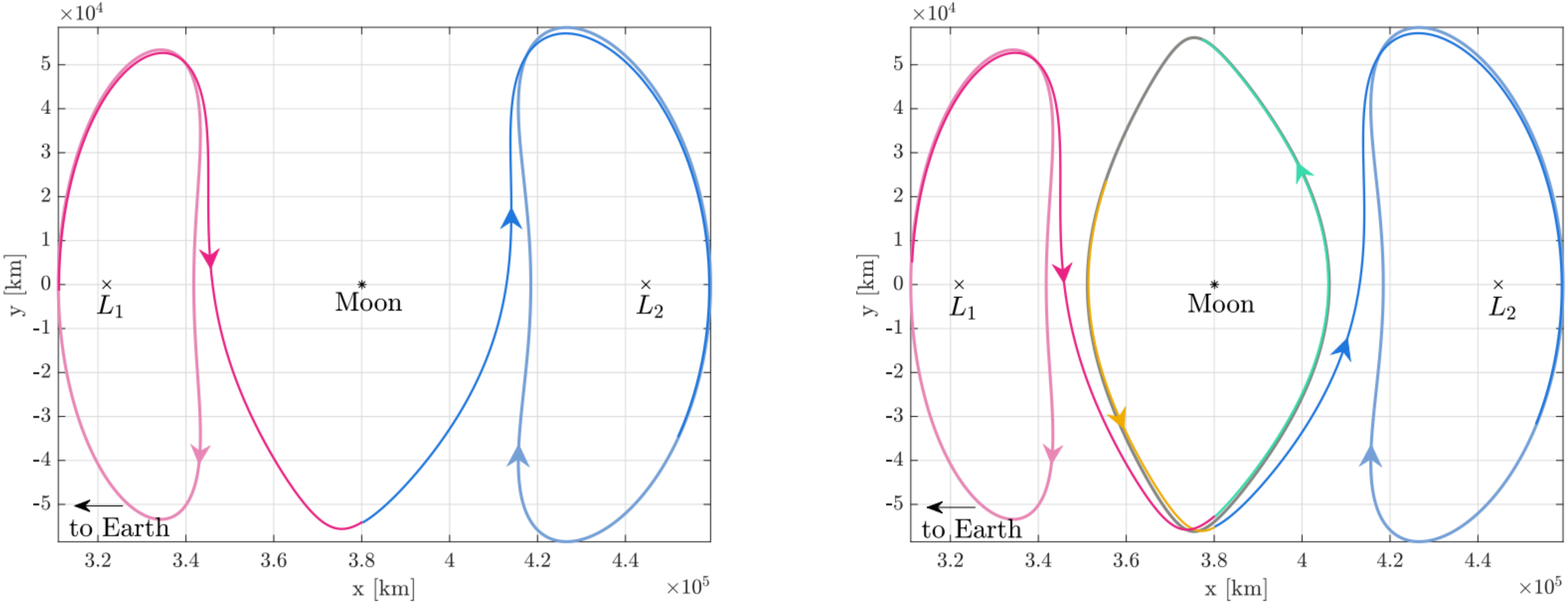


DPOs serve as heteroclinic link between L_1 and L_2 Lyapunov orbits

Gupta, Maaninee et al. "Earth-Moon multi-body orbits to facilitate cislunar surveillance activities." AIAA/AAS. (2021)

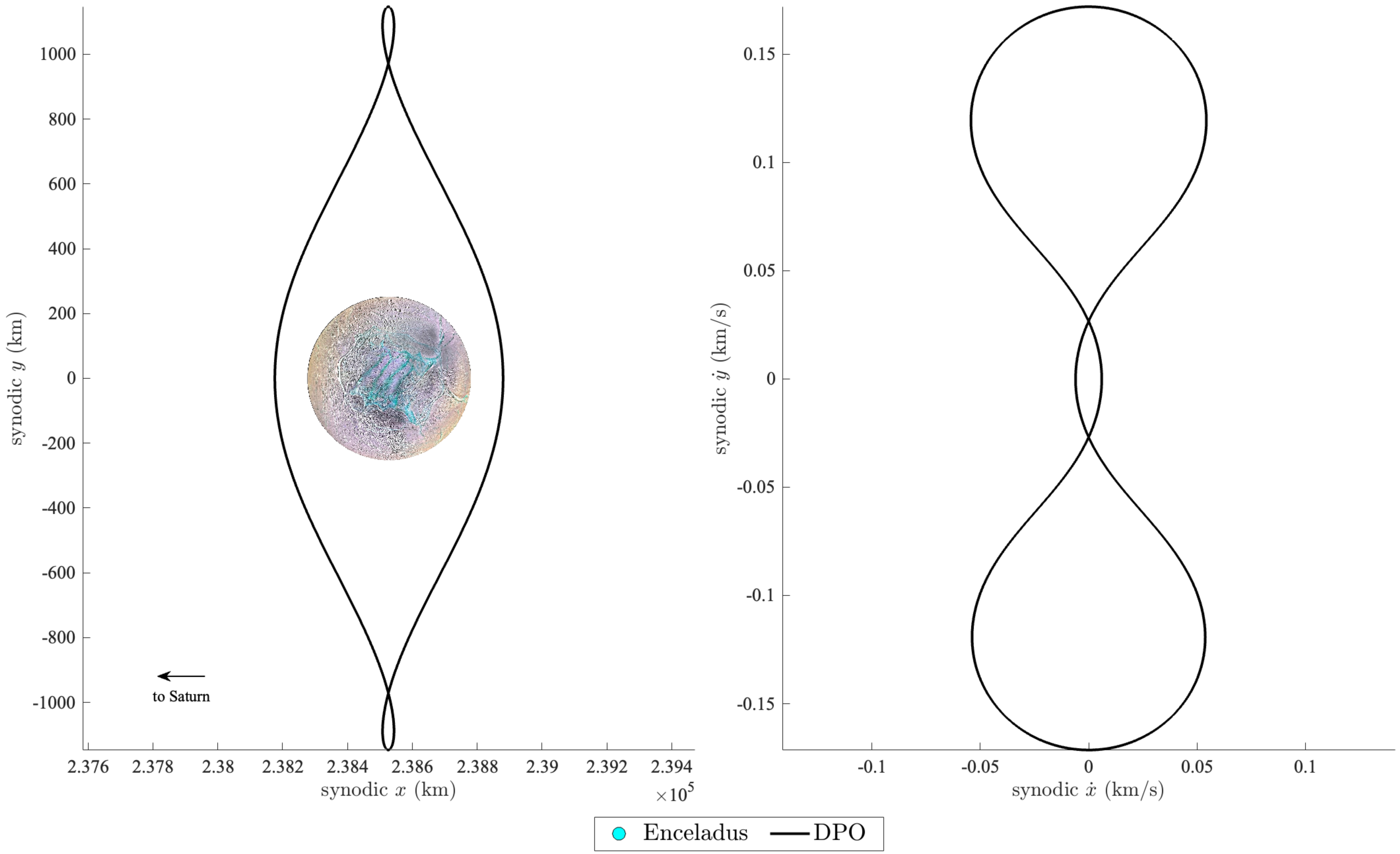
Astrodynamic Applications: Three-Body Problem

DPO uncertainty propagation conditions

- We choose an unstable DPO in the Saturn-Enceladus system for testing GBEES and other selected RBFs

$$\boldsymbol{x}_0 = \begin{bmatrix} 1.001471 & (\text{LU}) \\ -1.751810E-5 & (\text{LU}) \\ 0.0 & (\text{LU}) \\ 7.198783E-5 & (\text{LU/TU}) \\ 1.363392E-2 & (\text{LU/TU}) \\ 0.0 & (\text{LU/TU}) \end{bmatrix}$$

μ	1.9011E-7
LU (km)	238529
TU (s)	18913
T (hr)	19.5811
C	3.000078
SI	3.0187E+2



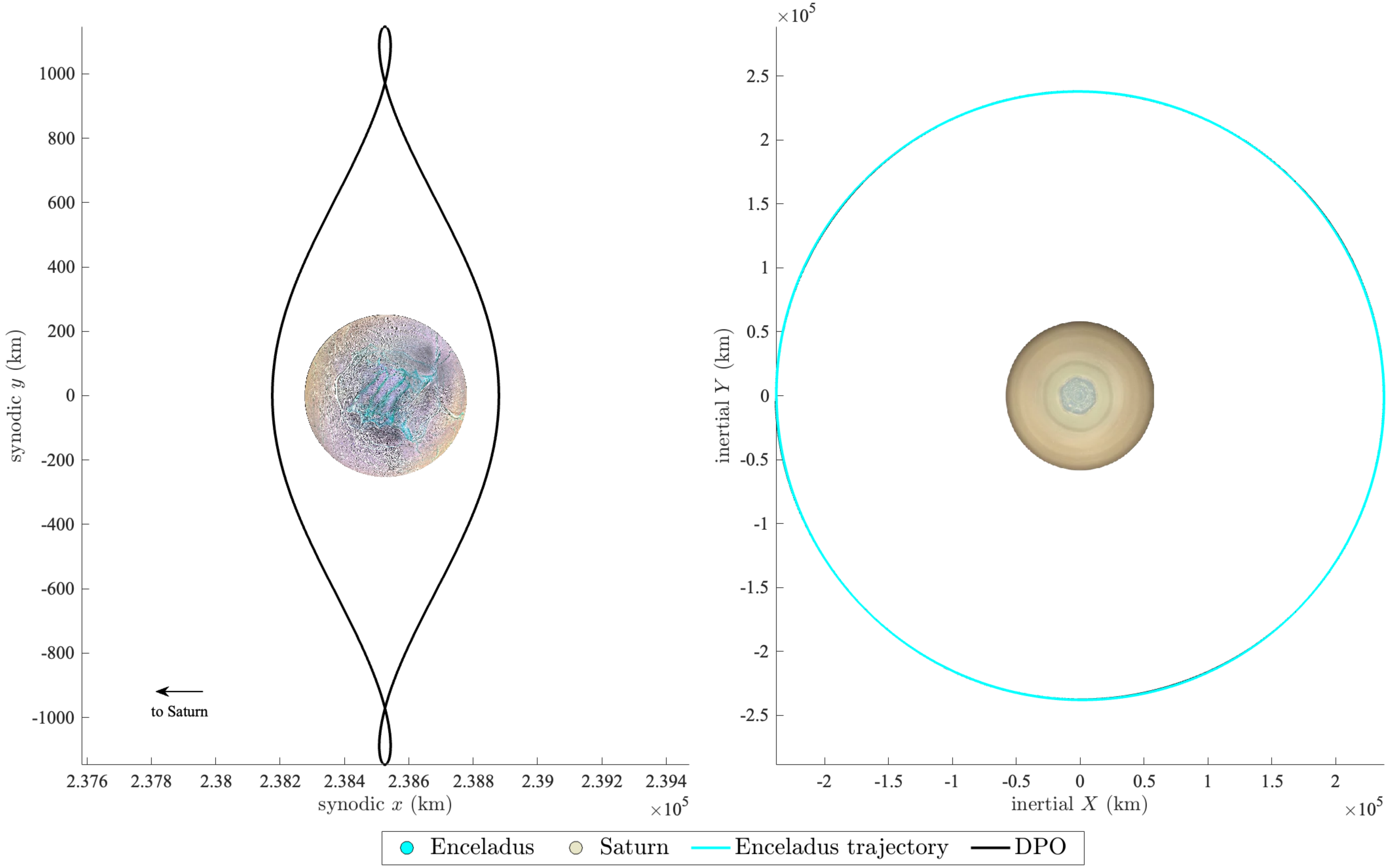
Astrodynamic Applications: Three-Body Problem

DPO uncertainty propagation conditions

- We choose an unstable DPO in the Saturn-Enceladus system for testing GBEES and other RBFs of note

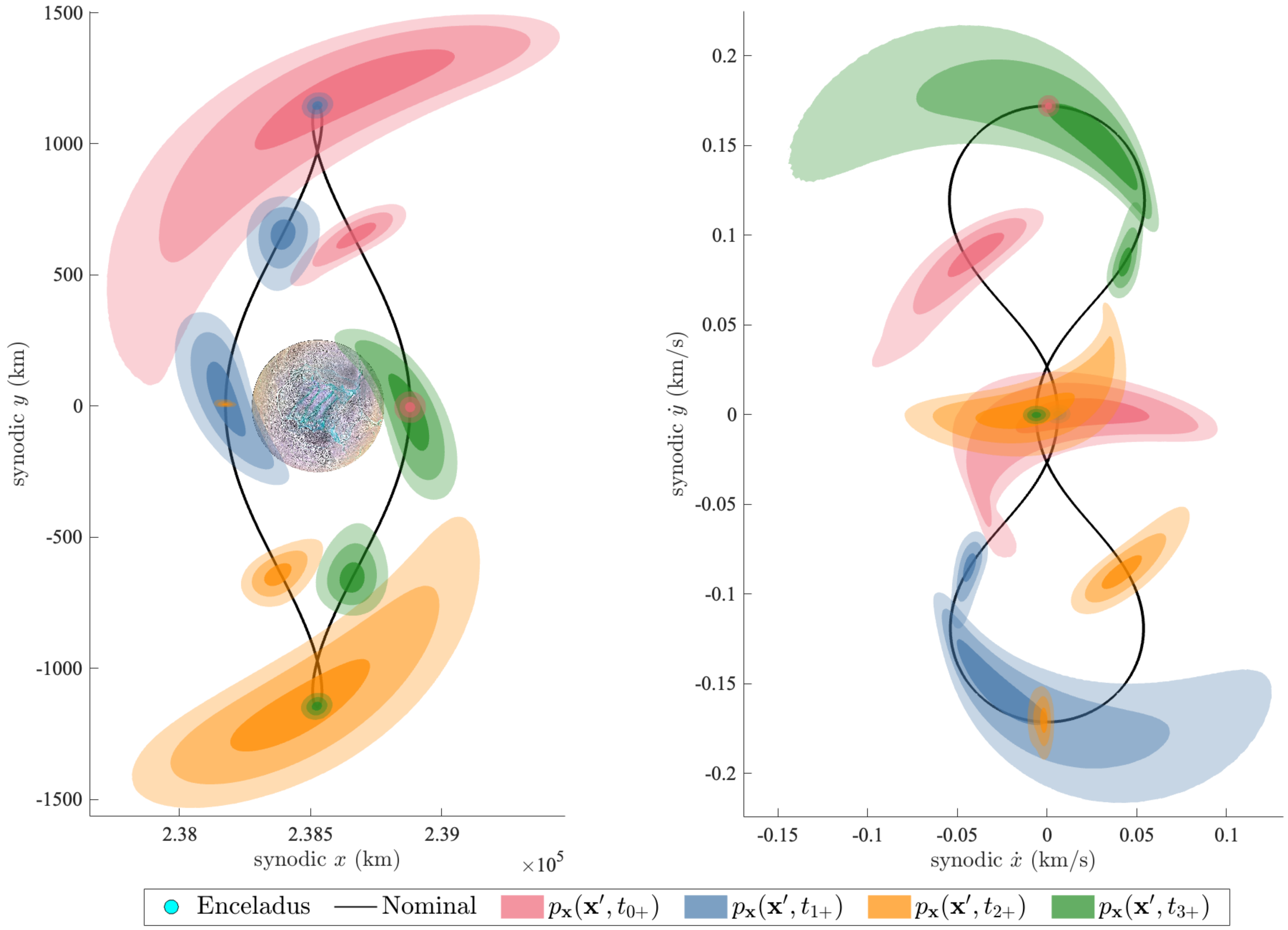
$$\boldsymbol{y} = \begin{bmatrix} \rho \\ \theta \\ \dot{\rho} \end{bmatrix} = \boldsymbol{h}(\boldsymbol{x}) = \begin{bmatrix} \sqrt{(x-1+\mu)^2 + y^2} \\ \tan^{-1}\left(\frac{y}{x-1+\mu}\right) \\ \frac{(x-1+\mu)\dot{x} + y\dot{y}}{\rho} \end{bmatrix}$$

σ_{ρ_0} (km)	20.0
σ_{θ_0} (rad)	1.74533E-2
$\sigma_{\dot{\rho}_0}$ (km/s)	2.0E-3
Δt_y (hr)	4.895



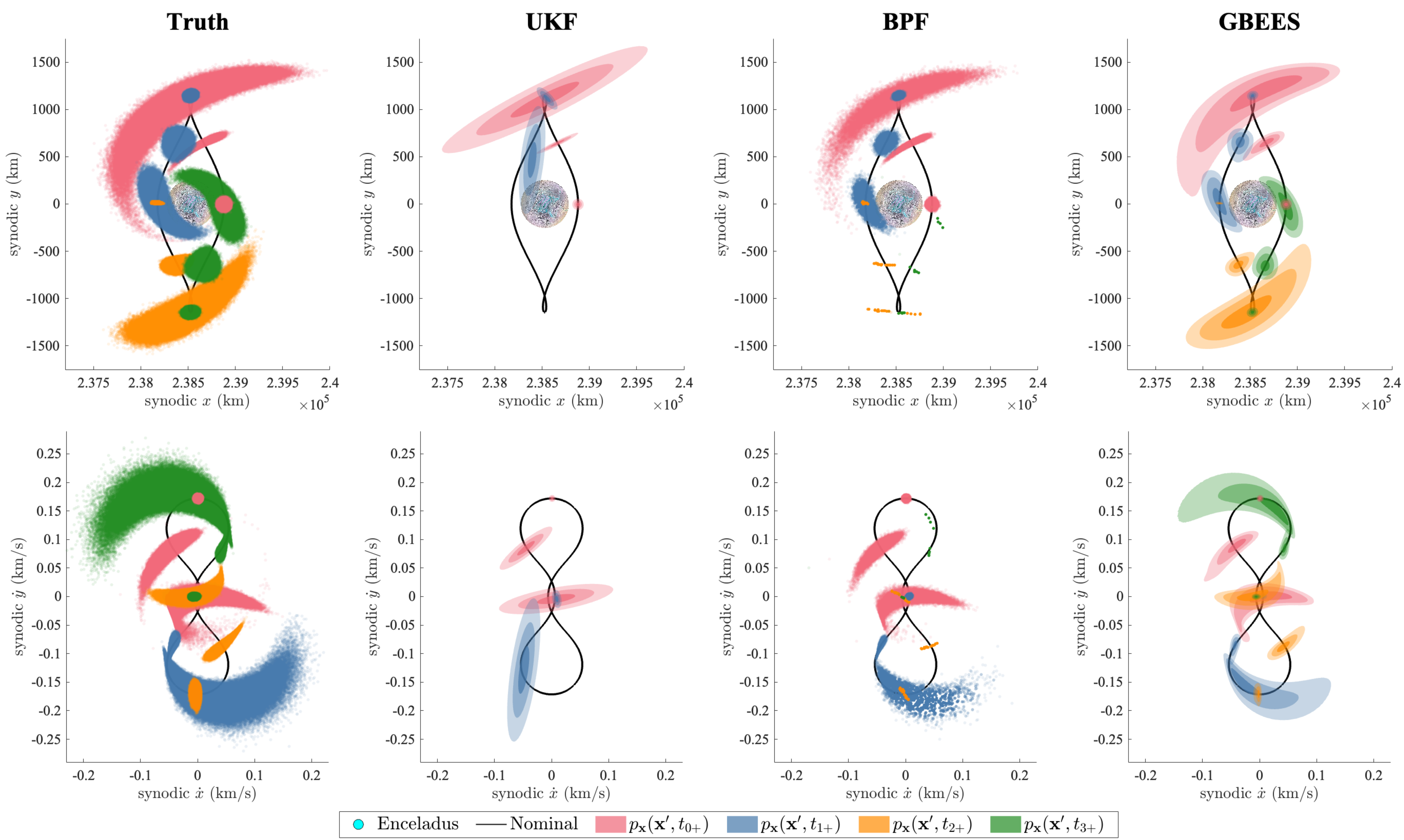
Saturn-Enceladus Distant Prograde Orbit Propagation

GBEES compared with other RBFs



Notes

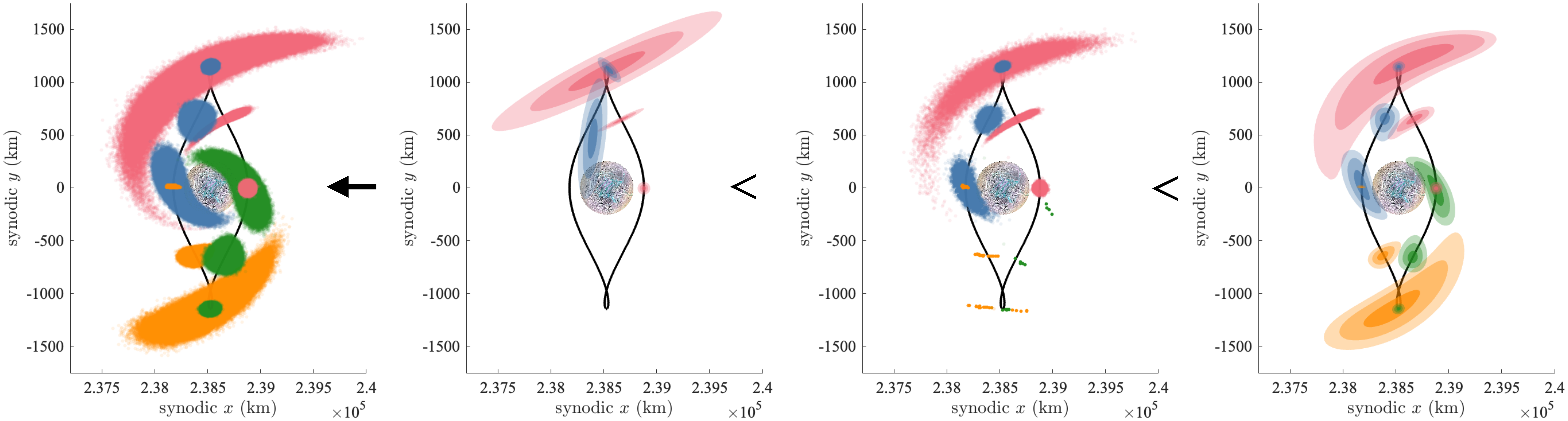
- Coordinates are in the synodic frame
- The true PDFs propagated by GBEES are 4D — these PDFS are the 4D ones integrated over velocity/position for visualization of the 2D position/velocity PDFs
- A change in color indicates a measurement update, with four occurring over this propagation period


Truth
UKF
BPF
GBEES
synodic y (km)
synodic x (km)
×10^5
synodic ẏ (km/s)
synodic ẋ (km/s)
Enceladus
Nominal
$p_{\mathbf{x}}(\mathbf{x}', t_{0+})$
$p_{\mathbf{x}}(\mathbf{x}', t_{1+})$
$p_{\mathbf{x}}(\mathbf{x}', t_{2+})$
$p_{\mathbf{x}}(\mathbf{x}', t_{3+})$

Saturn-Enceladus Distant Prograde Orbit Propagation

Non-Gaussian metric of comparison

- How do we compare the accuracy of these highly non-Gaussian distributions?



- A non-normal measure of the dissimilarity of distributions — **the Bhattacharyya Coefficient**

$$BC(P,Q) = \sum_{x \in \chi} \sqrt{P(\boldsymbol{x})\, Q(\boldsymbol{x})} \qquad \text{where} \qquad 0 \le BC(P,Q) \le 1$$

- $BC(P,Q) = 1$ indicates perfect overlap while $BC(P,Q) = 0$ indicates no overlap

Fukunaga, Keinosuke, Introduction to statistical pattern recognition. Elsevier. (2013)

Saturn-Enceladus Distant Prograde Orbit Propagation

Non-Gaussian metric of comparison

Algorithm 1 Calculate Bhattacharyya Coefficient of (P, Q)

Inputs: $P, Q : \mathbb{R}^d \to [0, 1],\ \ n > 1$

```
bounds ← [∞ · 1_{d×1}, −∞ · 1_{d×1}]
grid ← 0_{d×n}                                                          ▷ finding bounding region
for i = 1 to d do
    bounds[i, 1] ← min {b[i, 1], min{x[i] ∈ support(P ∪ Q)}}
    bounds[i, 2] ← max {b[i, 2], max{x[i] ∈ support(P ∪ Q)}}
    grid[i] ← {linear set from bounds[i, 1] to bounds[i, 2] of size n}
end for

χ ← 0_{n^d×d}
P*, Q* ← 0_{n^d×1}
count ← 1                                                               ▷ discretizing P, Q over χ
for x_1 = 1 to n do
    ⋮
    for x_d = 1 to n do
        χ[count] ← {grid[1, x_1], . . . , grid[d, x_d]}
        P*[count] ← P(χ[count])
        Q*[count] ← Q(χ[count])
        count ← count + 1
    end for
    ⋮
end for
P* ← P*/sum(P*)
Q* ← Q*/sum(Q*)

BC ← 0                                                                  ▷ calculating BC(P*, Q*)
for i = 1 to n^d do
    BC ← BC + √(P*[i]Q*[i])
end for
```

Saturn-Enceladus Distant Prograde Orbit Propagation

Non-Gaussian metric of comparison

Algorithm 1 Calculate Bhattacharyya Coefficient of (P, Q)

Inputs: $P, Q : \mathbb{R}^d \to [0,1],\ n > 1$

bounds $\leftarrow [\infty \cdot \mathbf{1}_{d\times 1}, -\infty \cdot \mathbf{1}_{d\times 1}]$
grid $\leftarrow \mathbf{0}_{d\times n}$ ▷ **finding bounding region**
for $i = 1$ to d **do**
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$\chi \leftarrow \mathbf{0}_{n^d \times d}$
$P^*, Q^* \leftarrow \mathbf{0}_{n^d \times 1}$
count $\leftarrow 1$ ▷ **discretizing** P, Q **over** χ
for $x_1 = 1$ to n **do**
  ⋮
  for $x_d = 1$ to n **do**
    χ[count] $\leftarrow$ {grid$[1, x_1], \ldots,$ grid$[d, x_d]$}
    P^*[count] $\leftarrow P(\chi$[count]$)$
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for $i = 1$ to n^d **do**
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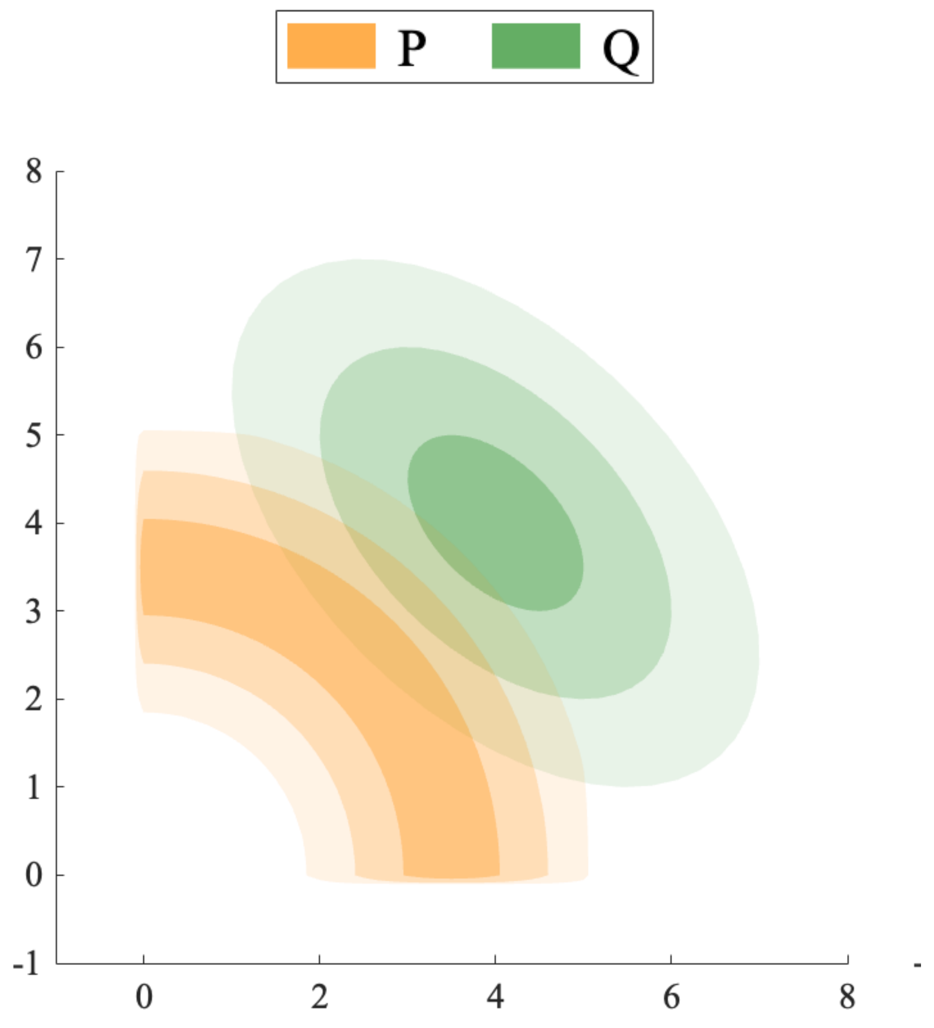
end for



$n = 10$

Saturn-Enceladus Distant Prograde Orbit Propagation

Non-Gaussian metric of comparison

Algorithm 1 Calculate Bhattacharyya Coefficient of (P, Q)

Inputs: $P, Q : \mathbb{R}^d \to [0, 1], \; n > 1$

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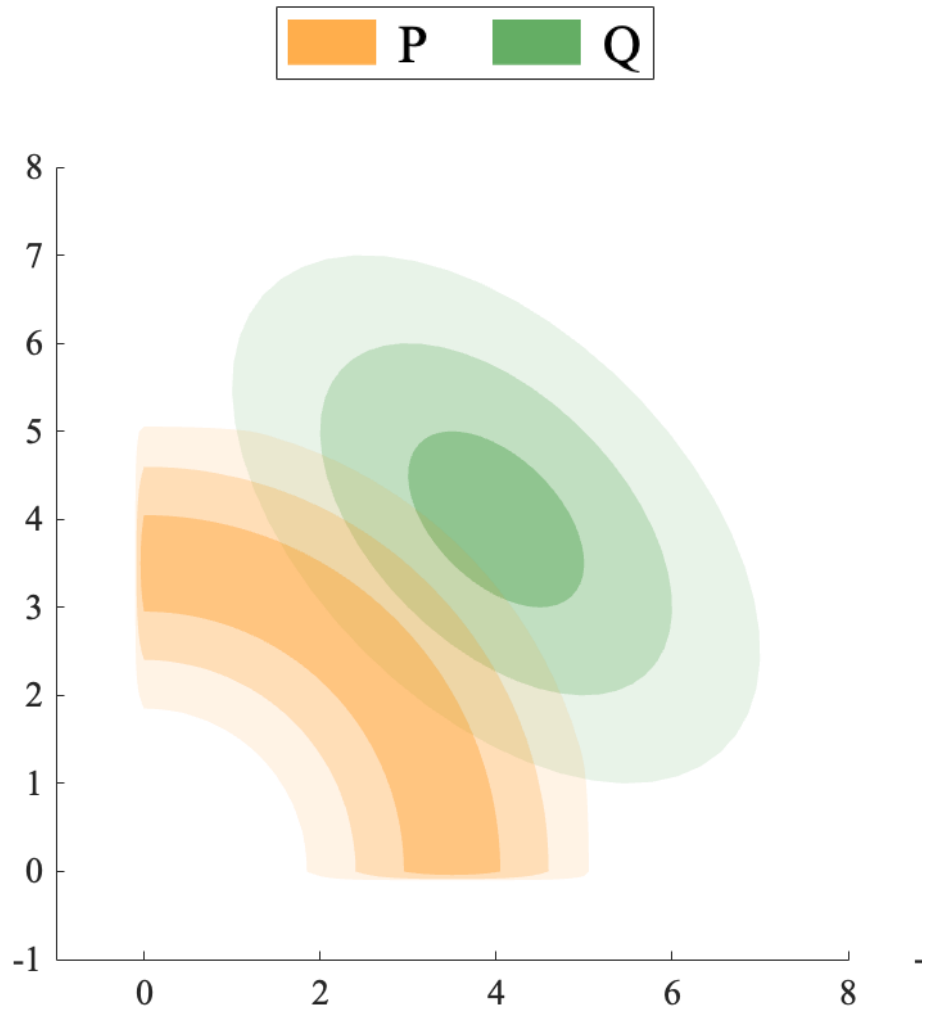
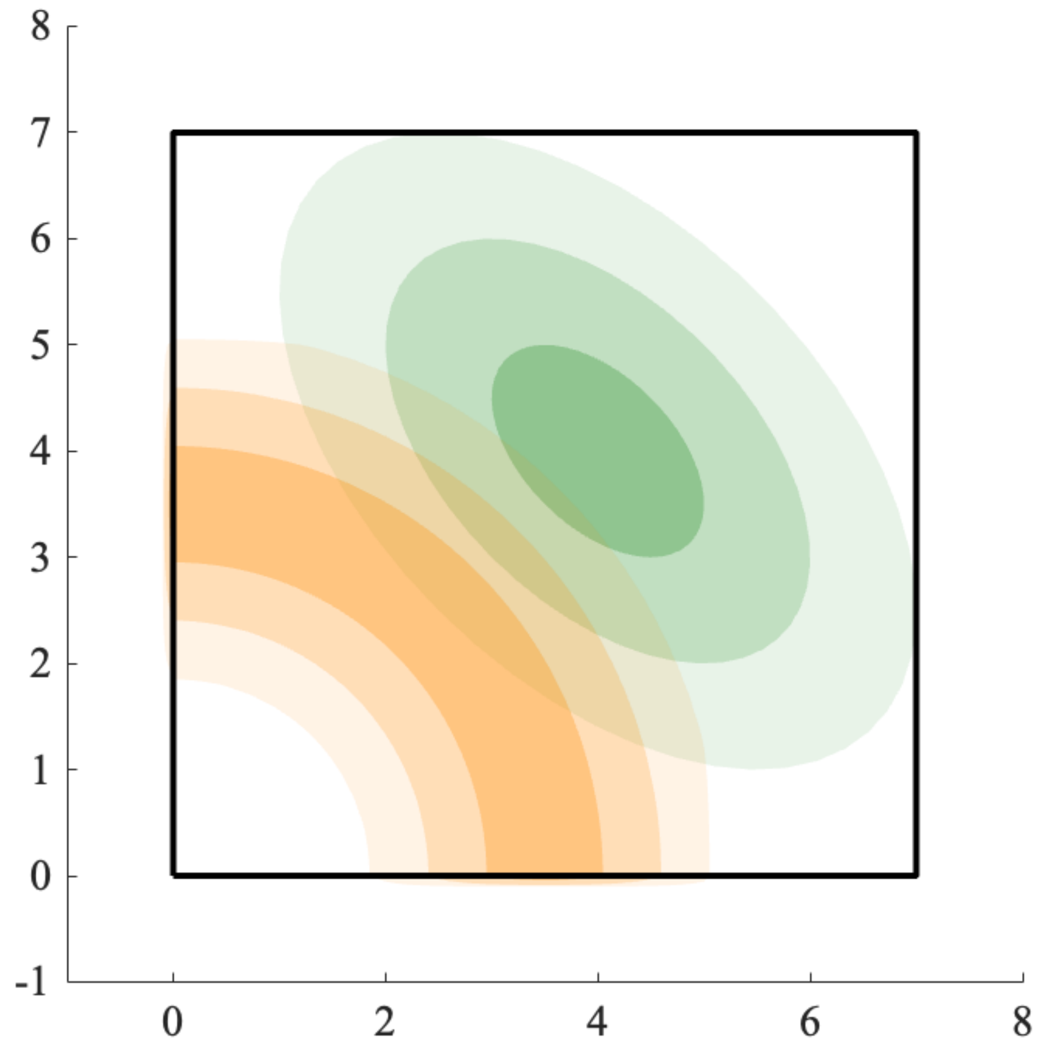


$n = 10$



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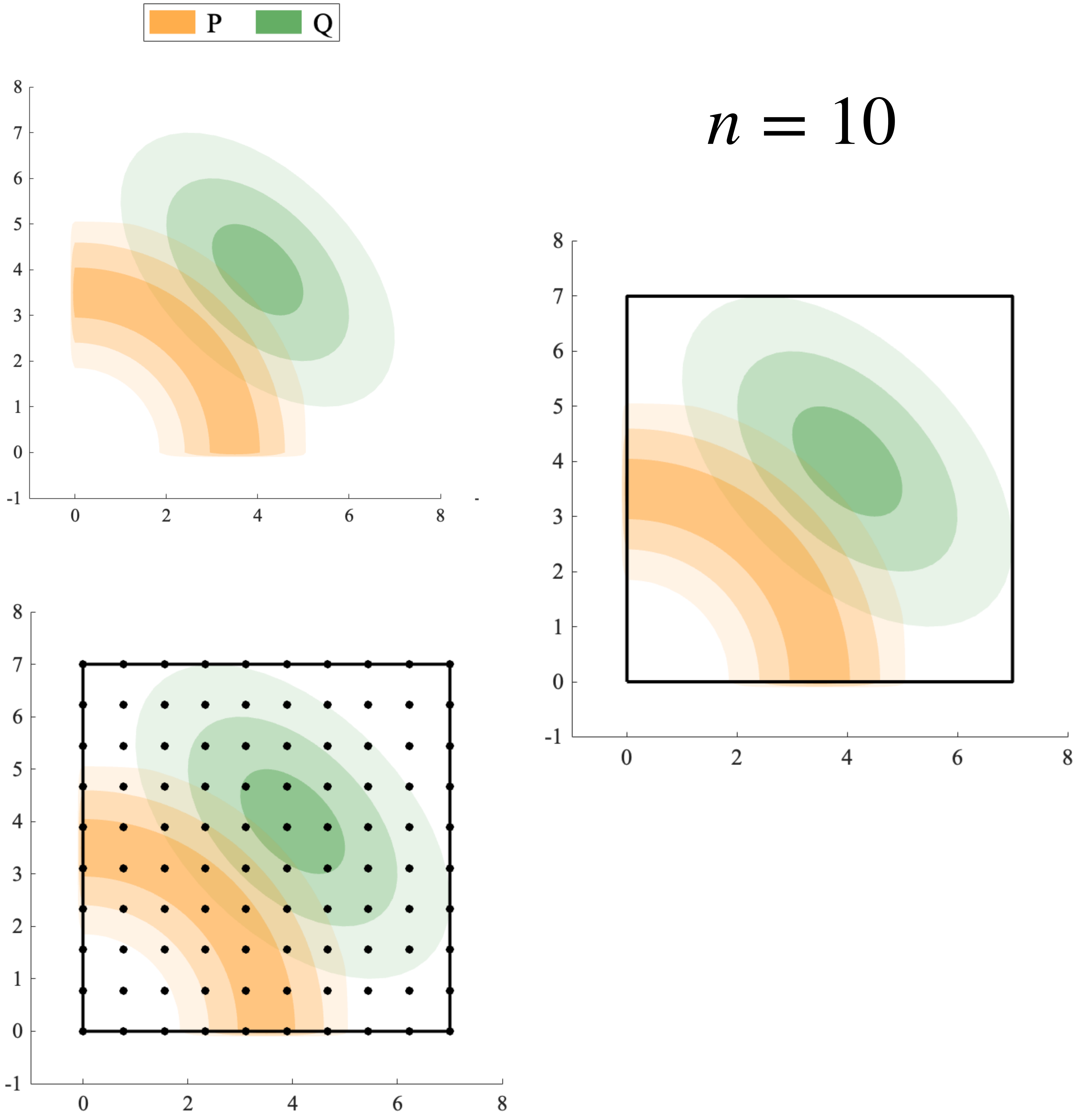

Saturn-Enceladus Distant Prograde Orbit Propagation

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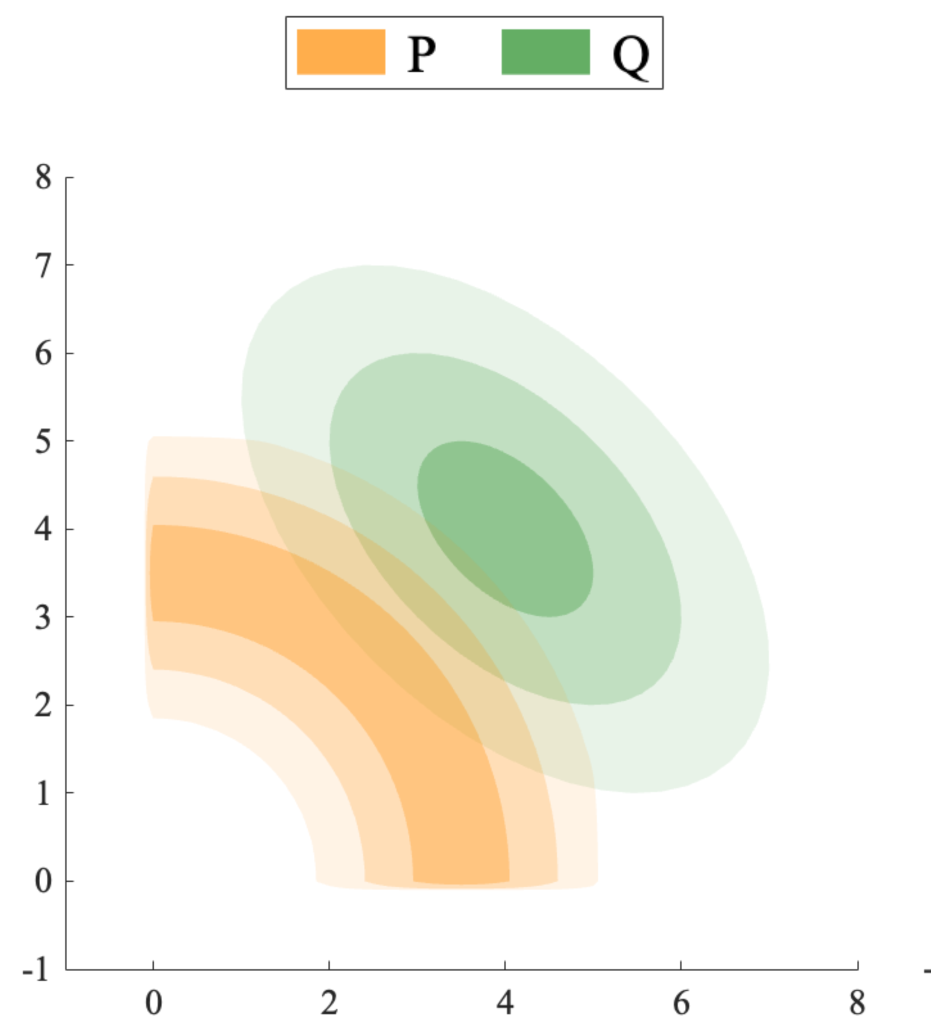
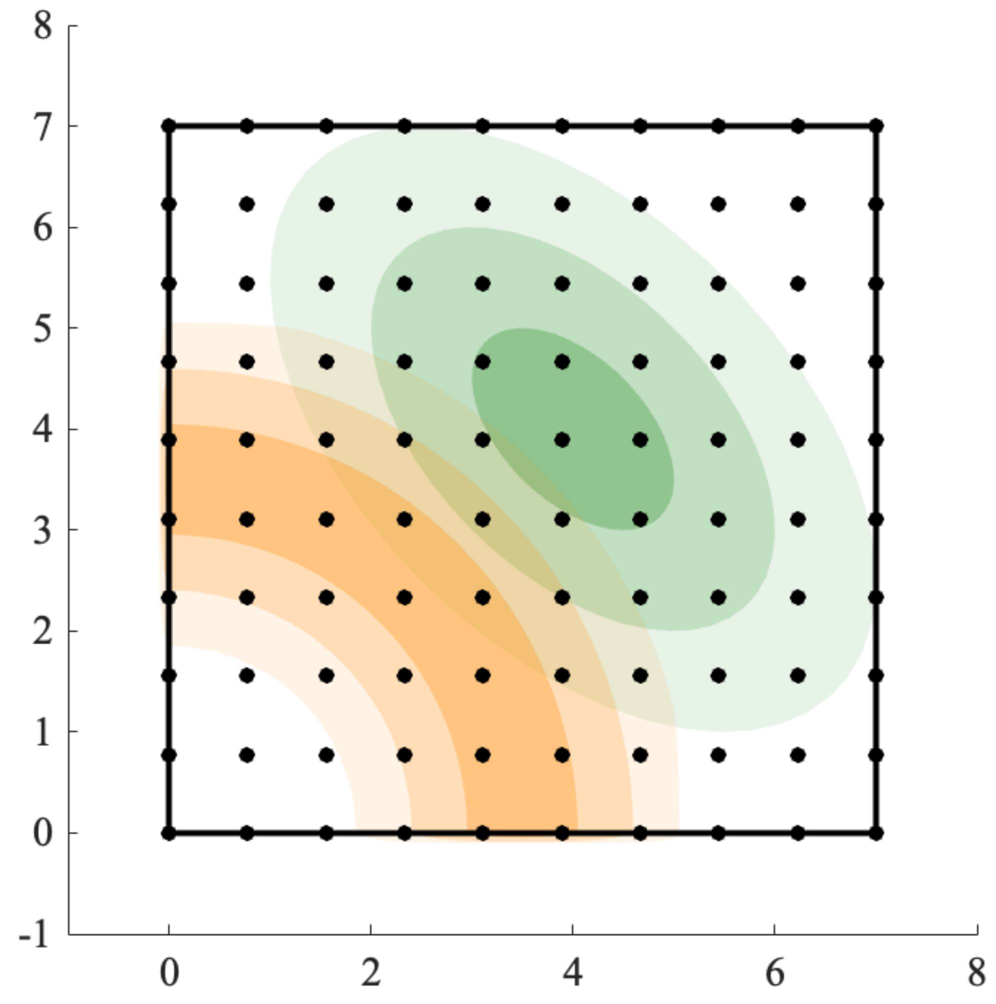
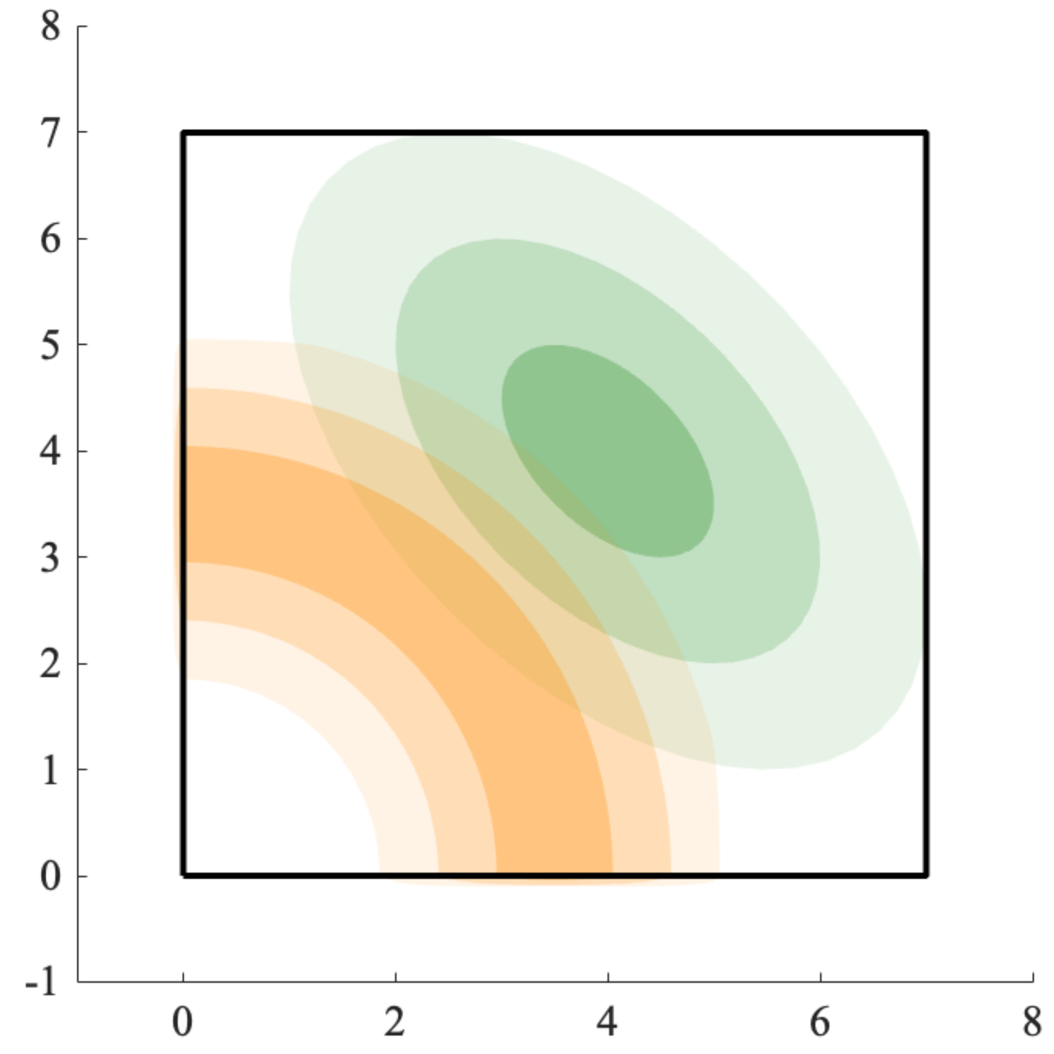




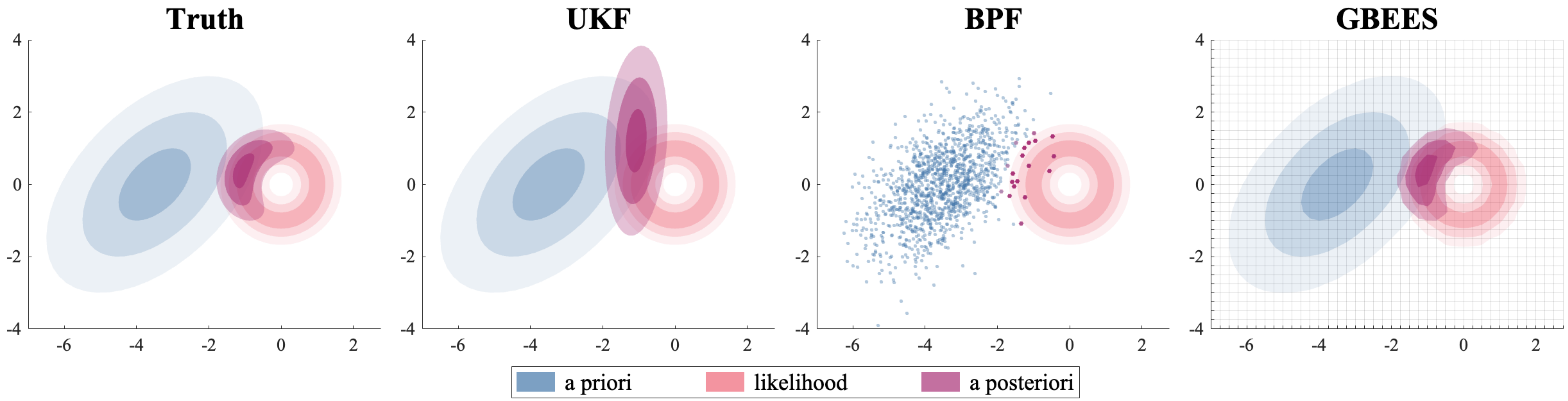
$n = 10$



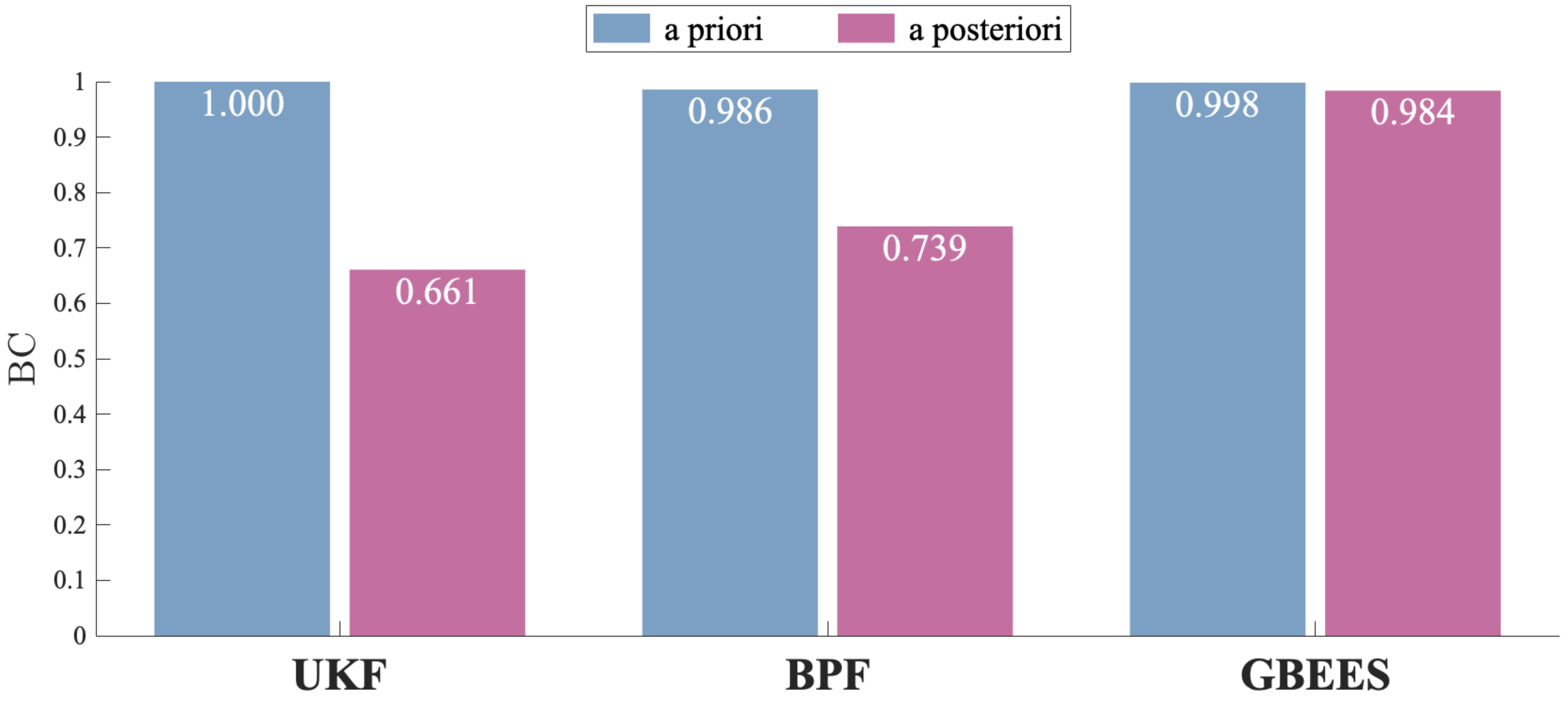
$BC(P, Q) = 0.1762$

Saturn-Enceladus Distant Prograde Orbit Propagation

Non-Gaussian metric of comparison

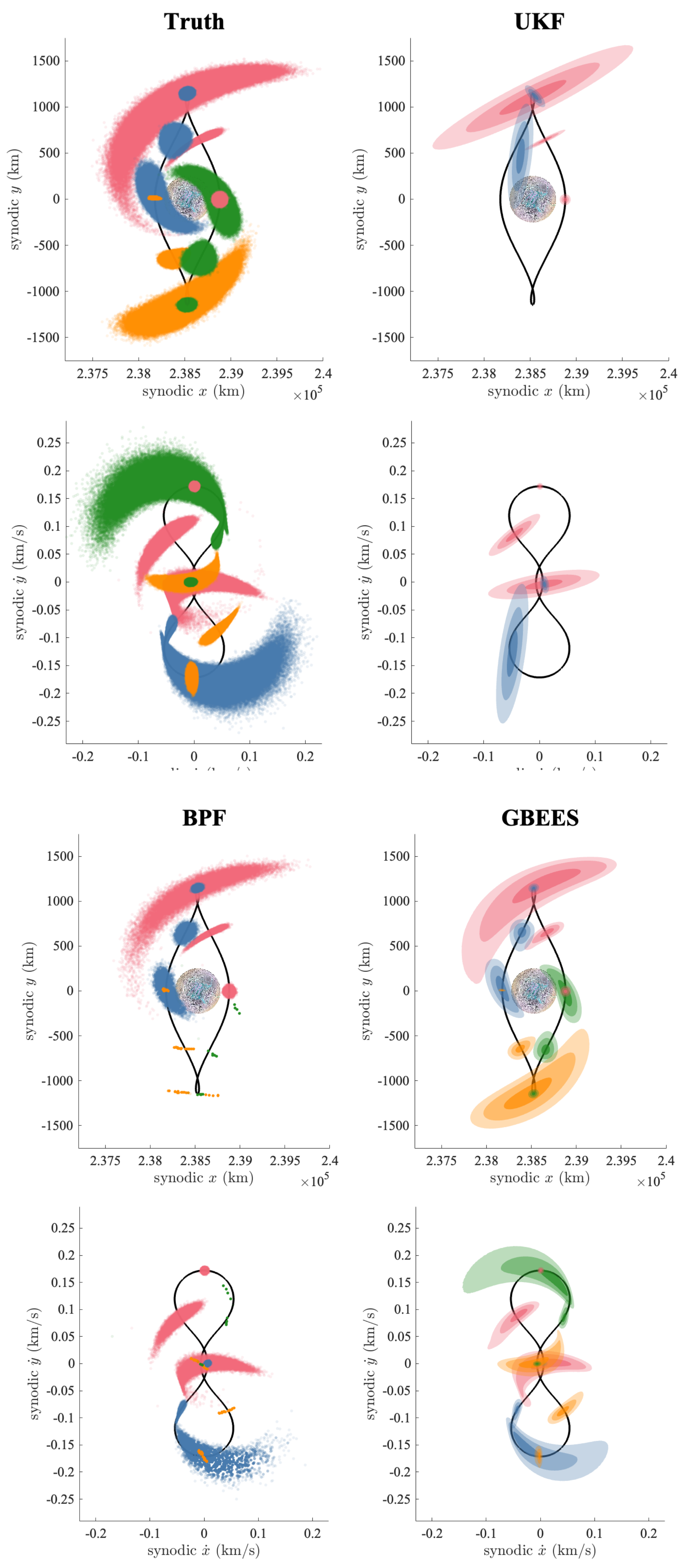


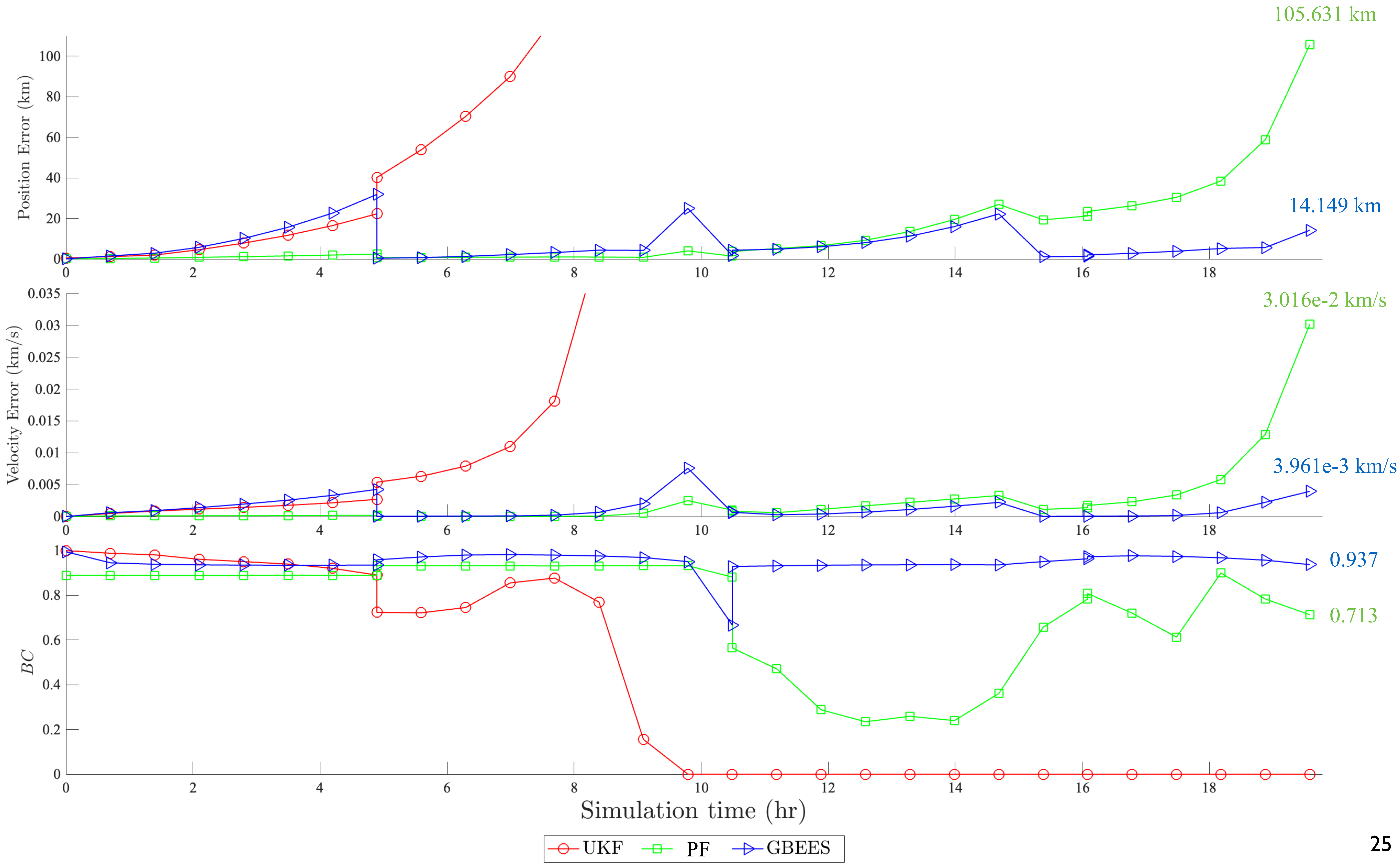
- As expected, GBEES performs best in this 2D test problem
- The GBEES update step is just the truth update step, just at a lower refinement



Saturn-Enceladus Distant Prograde Orbit Propagation

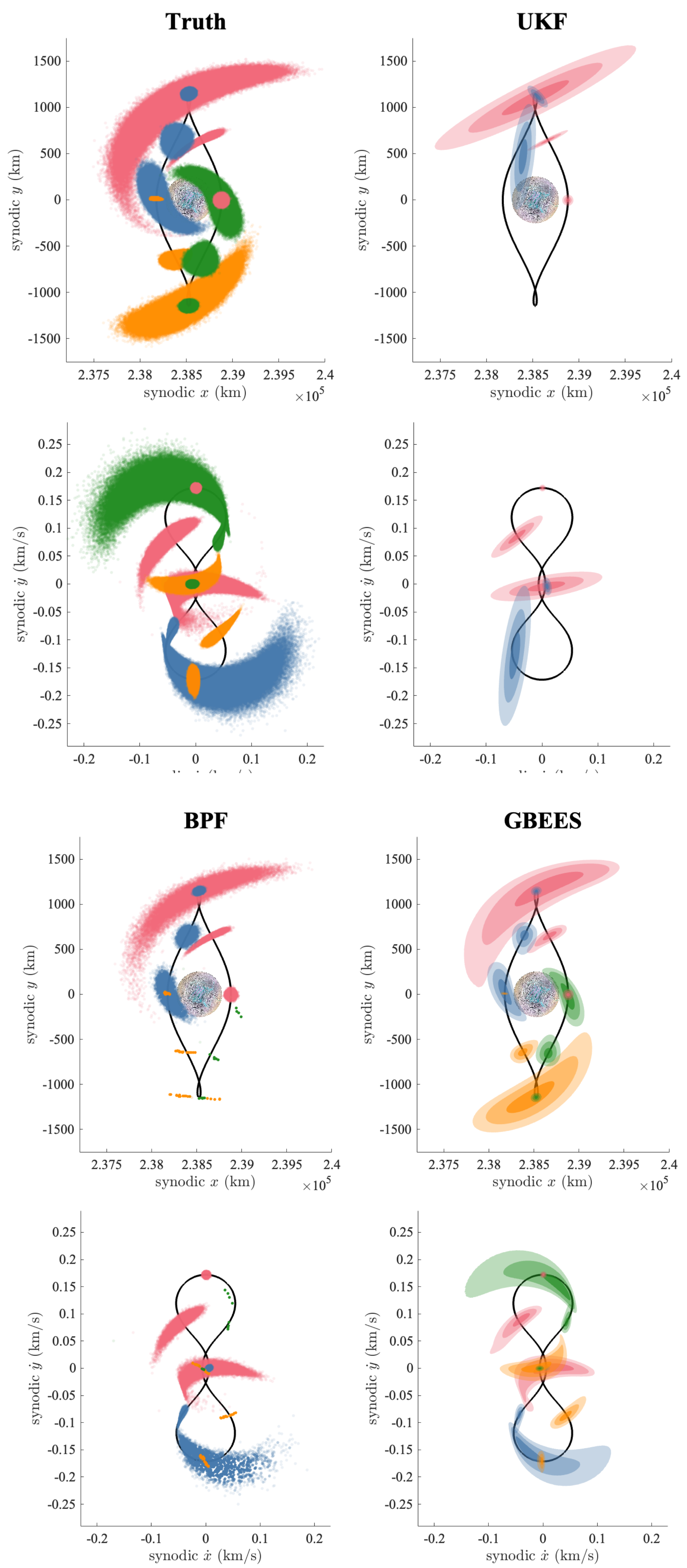
Quantitative comparison

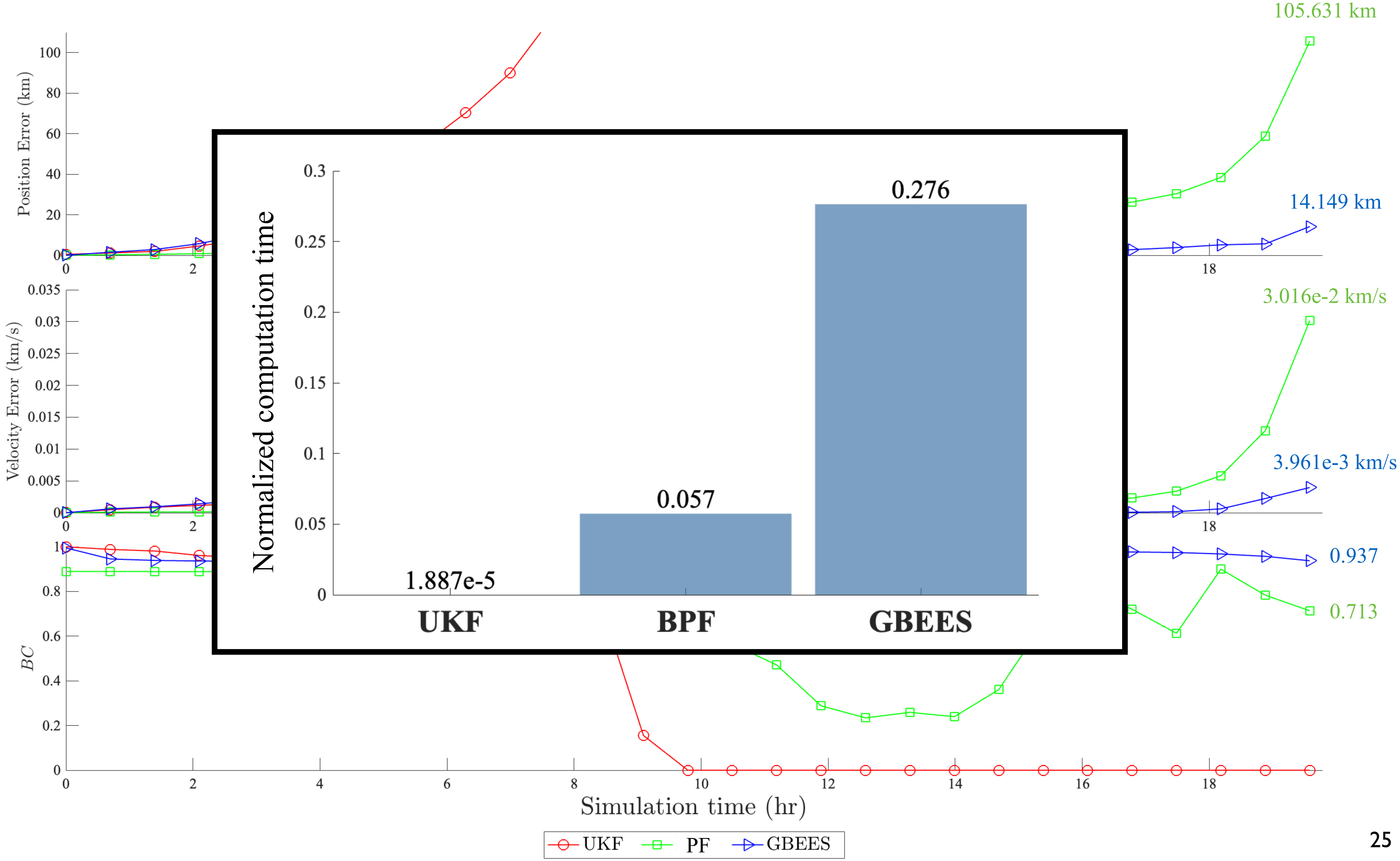




Saturn-Enceladus Distant Prograde Orbit Propagation

Quantitative comparison





Goal: Embed Monte within GBEES for ephemeris-quality orbital uncertainty propagation

Monte Python Wrapper

- For computational efficiency, GBEES runs in C — embedding Monte within GBEES requires a **Python wrapper**

GBEES

filetype = .c

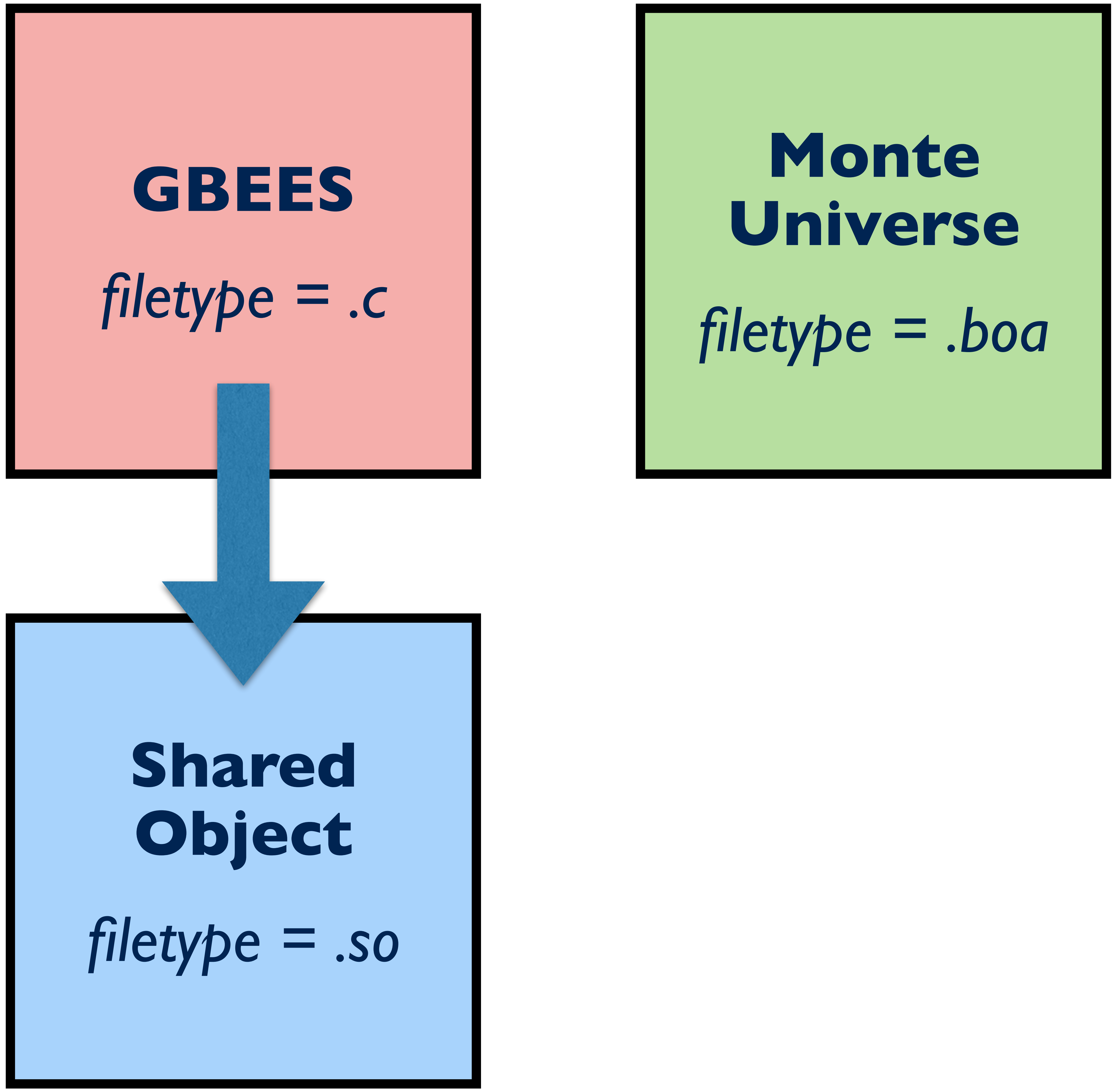
Monte Universe

filetype = .boa

<u>Order of Operations</u>

Monte Python Wrapper

- For computational efficiency, GBEES runs in C — embedding Monte within GBEES requires a **Python wrapper**



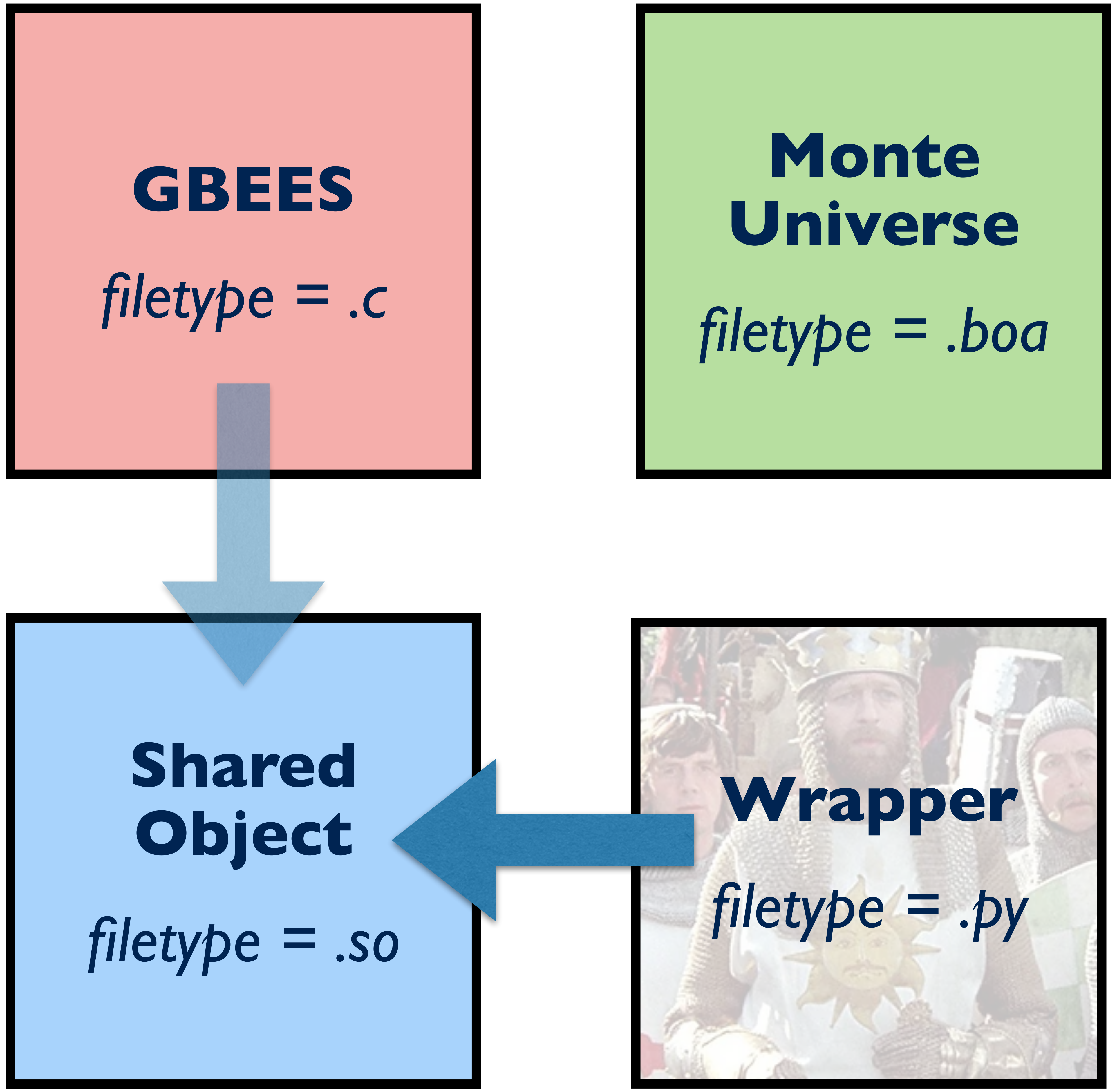
Order of Operations

1. Compile *GBEES.c* with Monte to a shared object

```
$ mdock run gcc -shared -o GBEES.so GBEES.c
```

Monte Python Wrapper

- For computational efficiency, GBEES runs in C — embedding Monte within GBEES requires a **Python wrapper**



Order of Operations

1. Compile *GBEES.c* with Monte to a shared object

```
$ mdock run gcc -shared -o GBEES.so GBEES.c
```

2. Dynamically link *Wrapper.py* to *GBEES.so*

```
>> lib = ctypes.CDLL(“GBEES.so")
```

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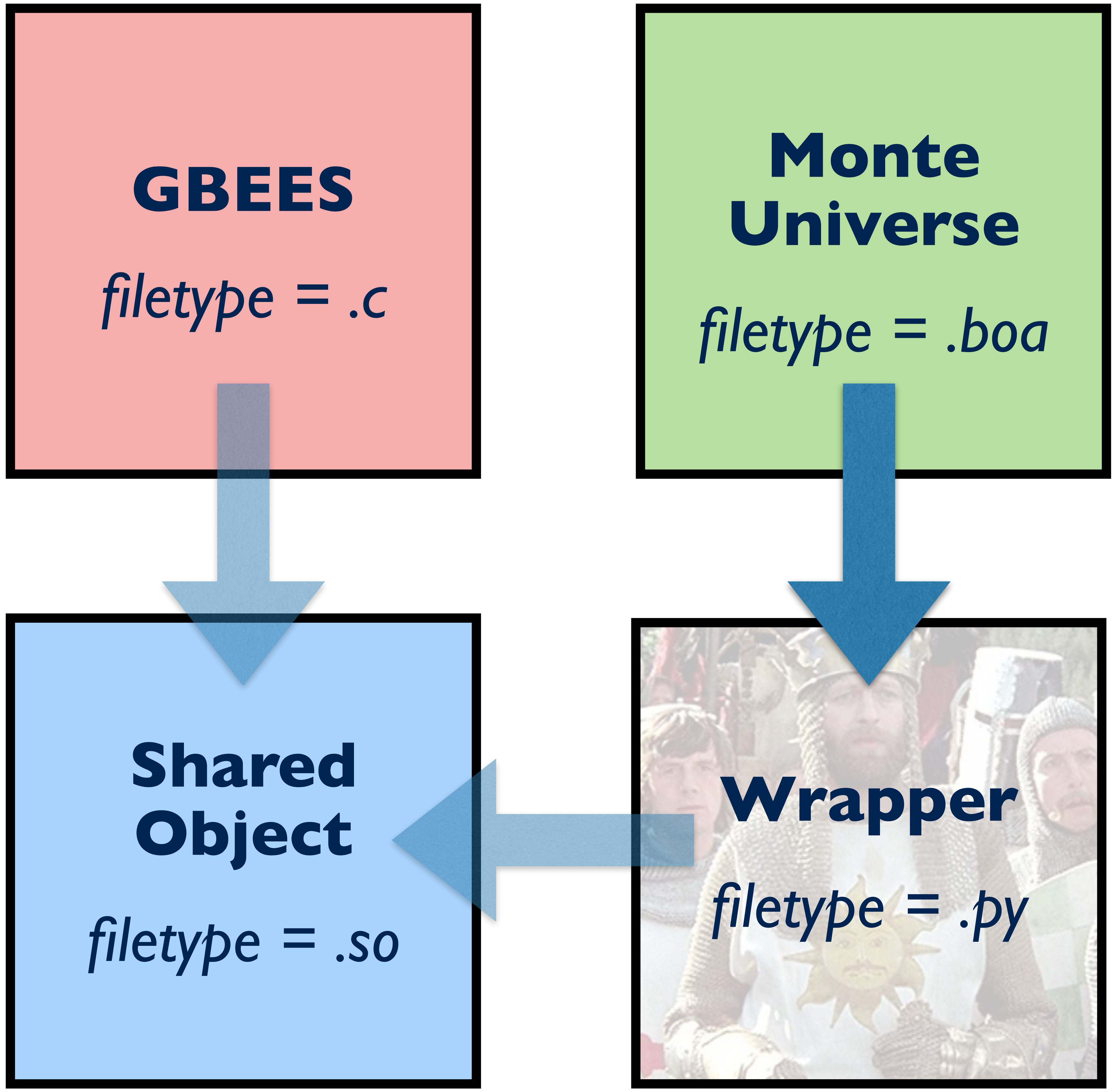

Order of Operations

1. Compile *GBEES.c* with Monte to a shared object

```
$ mdock run gcc -shared -o GBEES.so GBEES.c
```

2. Dynamically link *Wrapper.py* to *GBEES.so*

```
>> lib = ctypes.CDLL("GBEES.so")
```

3. Pass *MonteUniverse.boa* to *Wrapper.py*

```
>> boa = Monte.BoaLoad("MonteUniverse.boa")
```

Monte Python Wrapper

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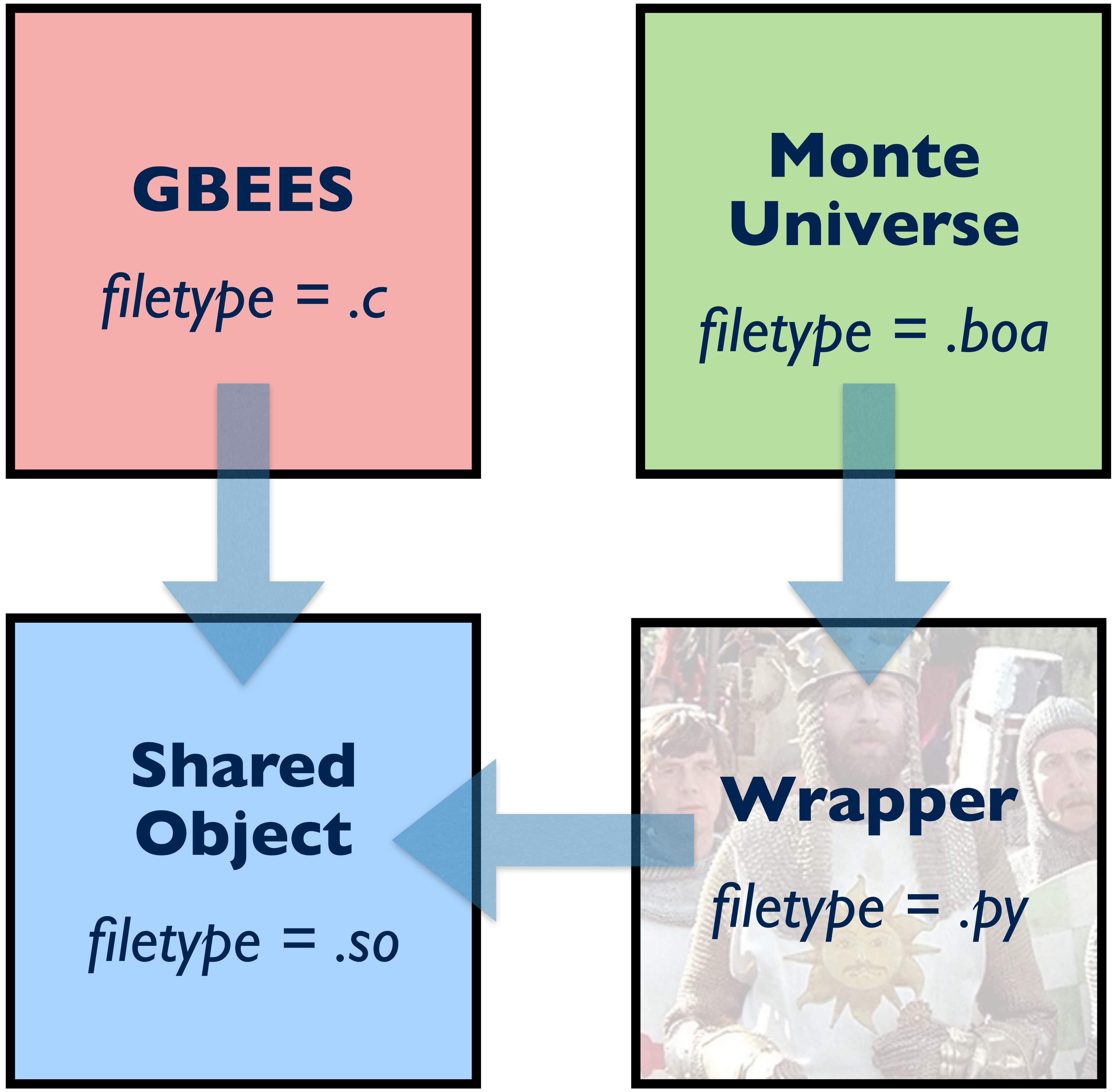

Order of Operations

1. Compile *GBEES.c* with Monte to a shared object

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>> lib = ctypes.CDLL(“GBEES.so")
```

3. Pass *MonteUniverse.boa* to *Wrapper.py*

```
>> boa = Monte.BoaLoad(“MonteUniverse.boa”)
```

4. Run GBEES with Monte by passing the *.boa* to the linked library

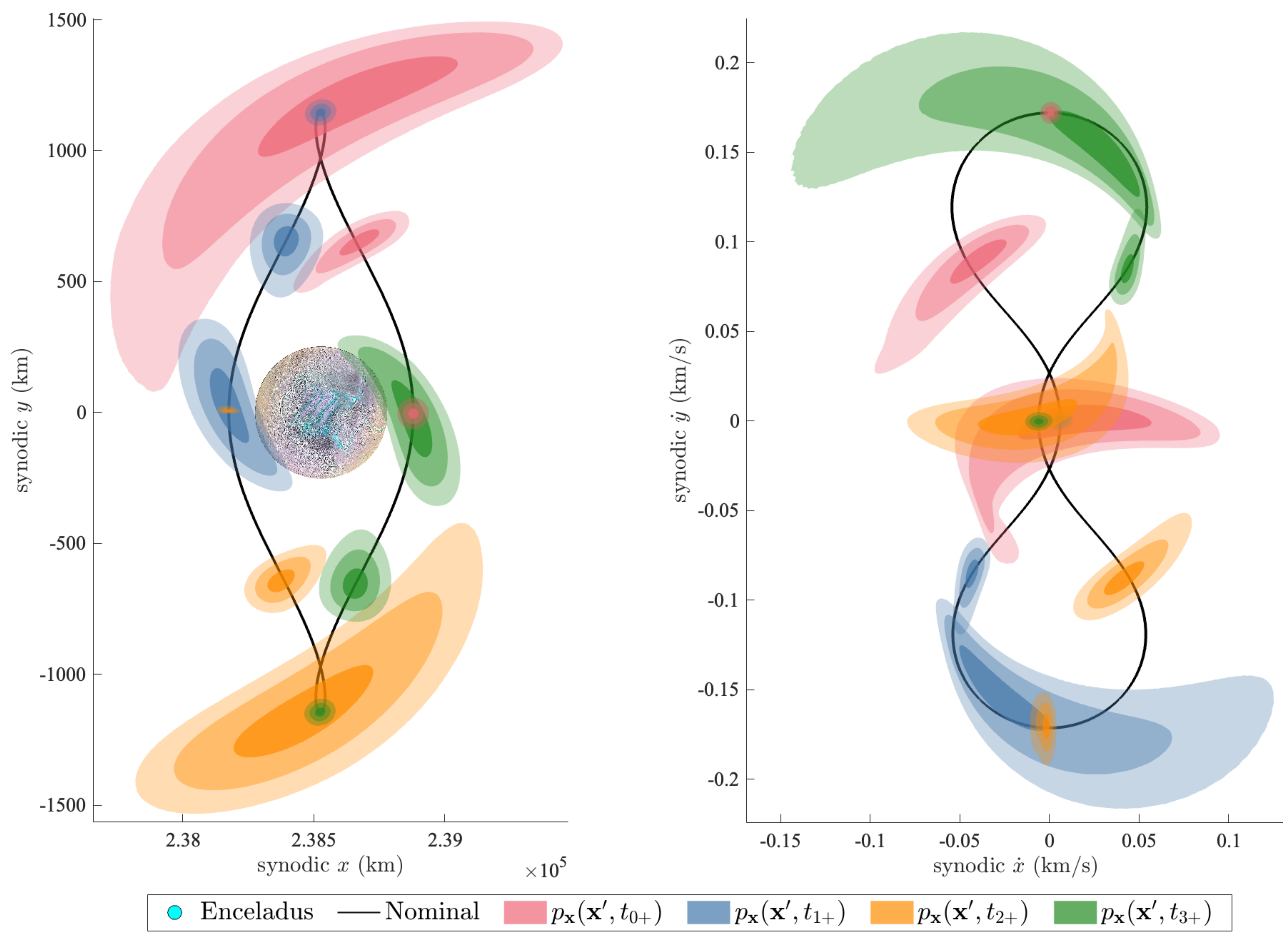
```
>> lib.run_gbees(boa)
```

Saturn-Enceladus Distant Prograde Orbit Propagation

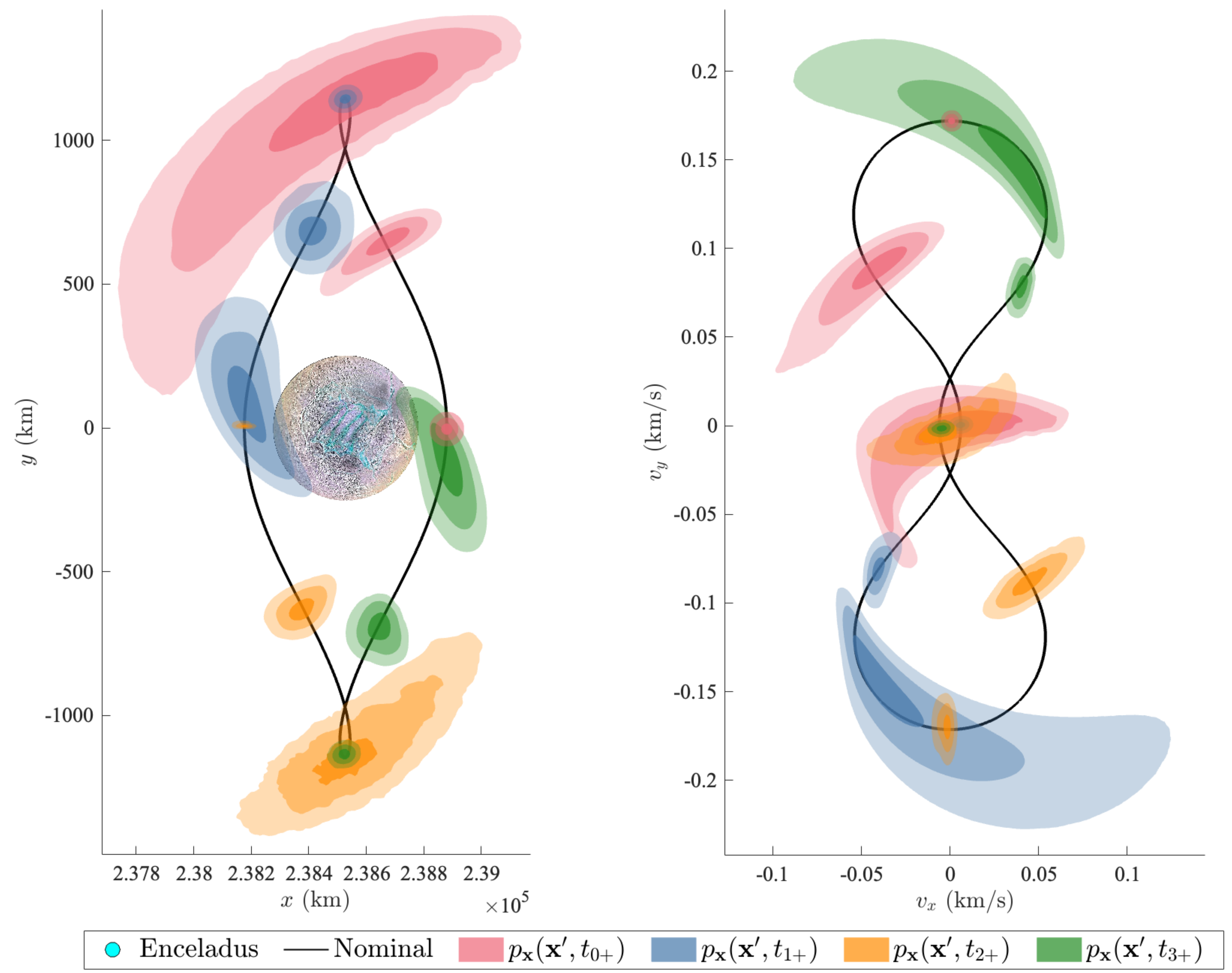
Monte vs. Analytical comparison - accuracy

- We compare the **PDFs** when propagating GBEES with the analytical solution to the CR3BP vs. when propagating GBEES with dynamics sourced from Monte

Analytical



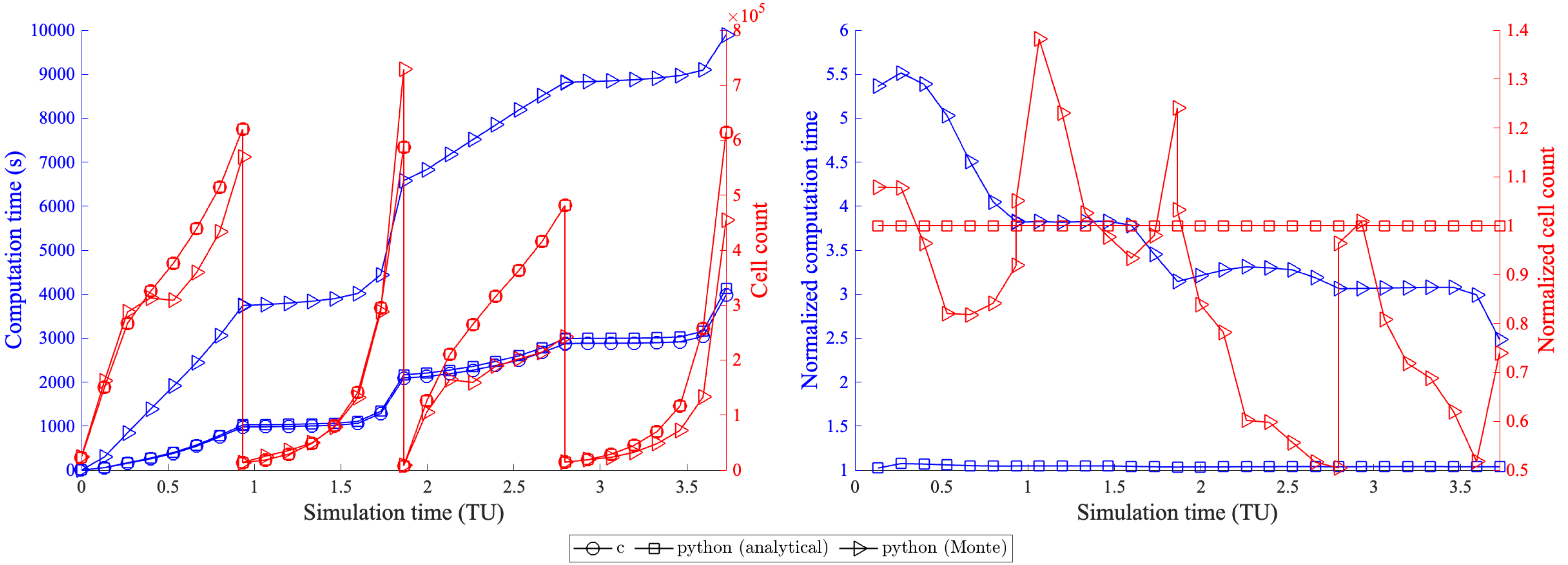
Monte



Saturn-Enceladus Distant Prograde Orbit Propagation

Monte vs. Analytical comparison - efficiency

- We compare the **computation time** when propagating GBEES with the analytical solution to the CR3BP vs. when propagating GBEES with dynamics sourced from Monte

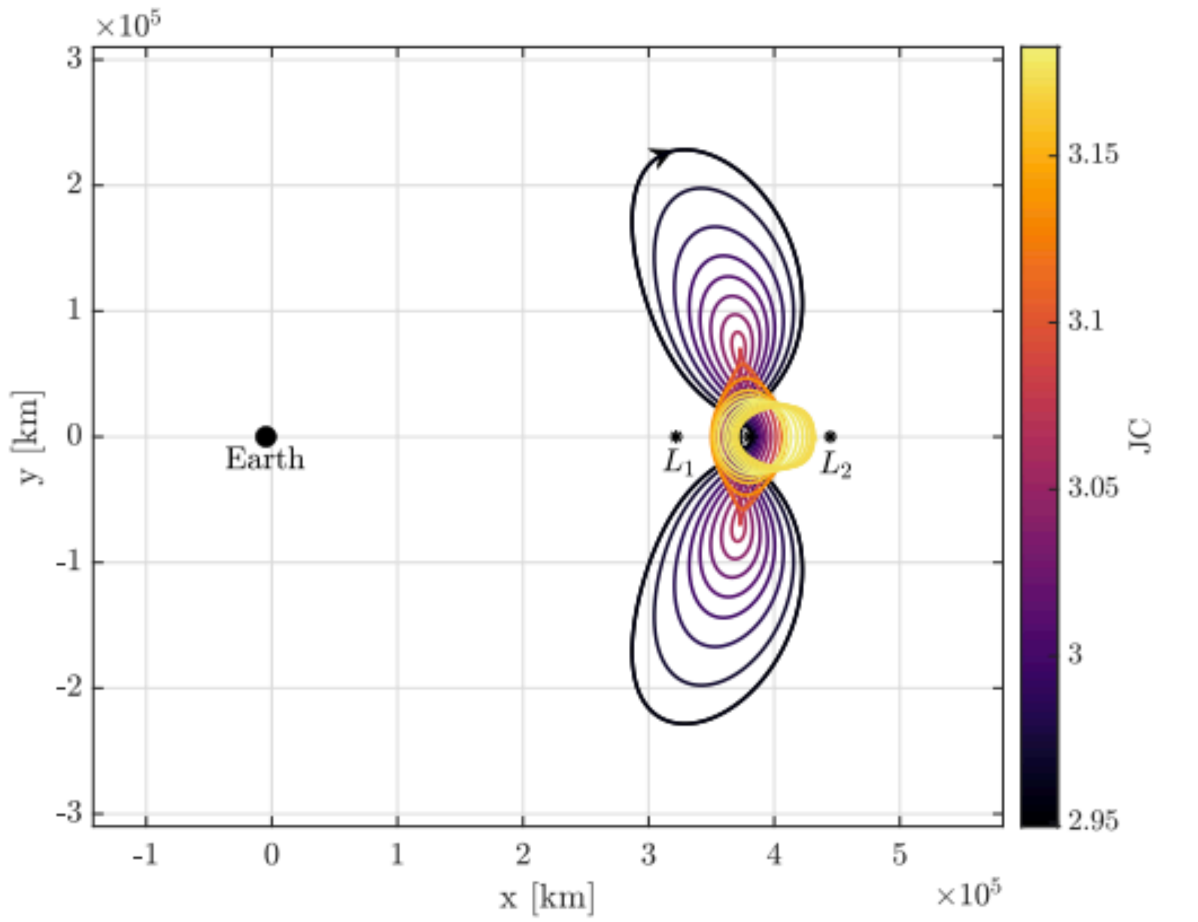


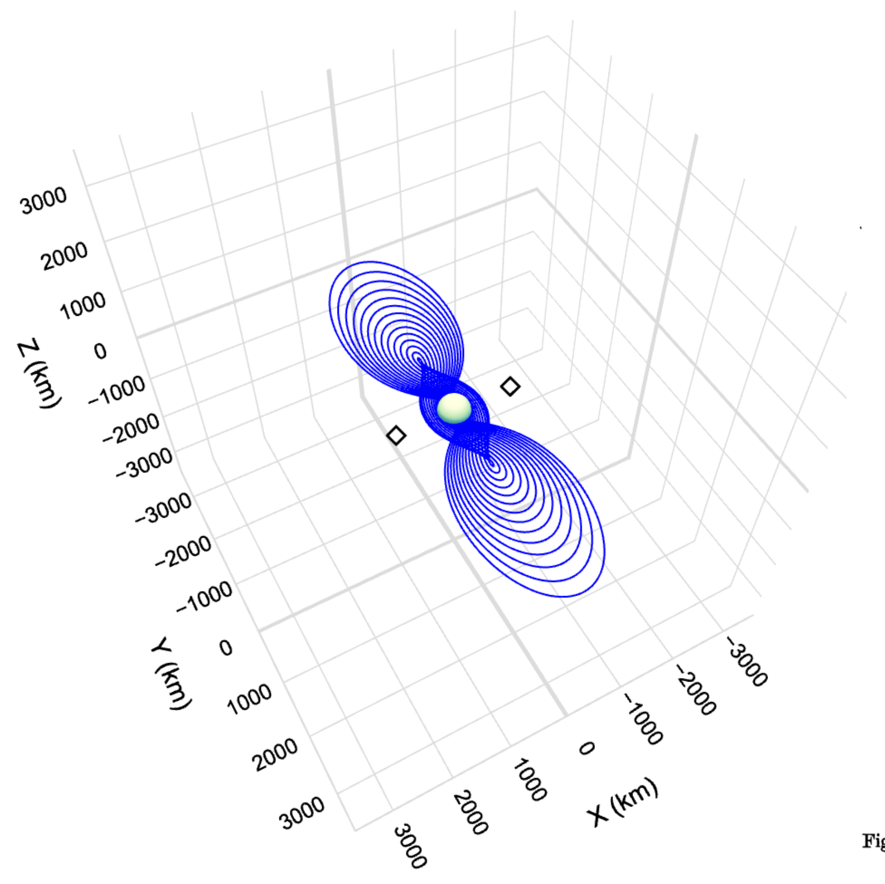
- There are still some bugs to fix here!

Conclusions

Conclusions

- There exist **favorable trajectories** in deep space where uncertainty may **realistically** become non-Gaussian





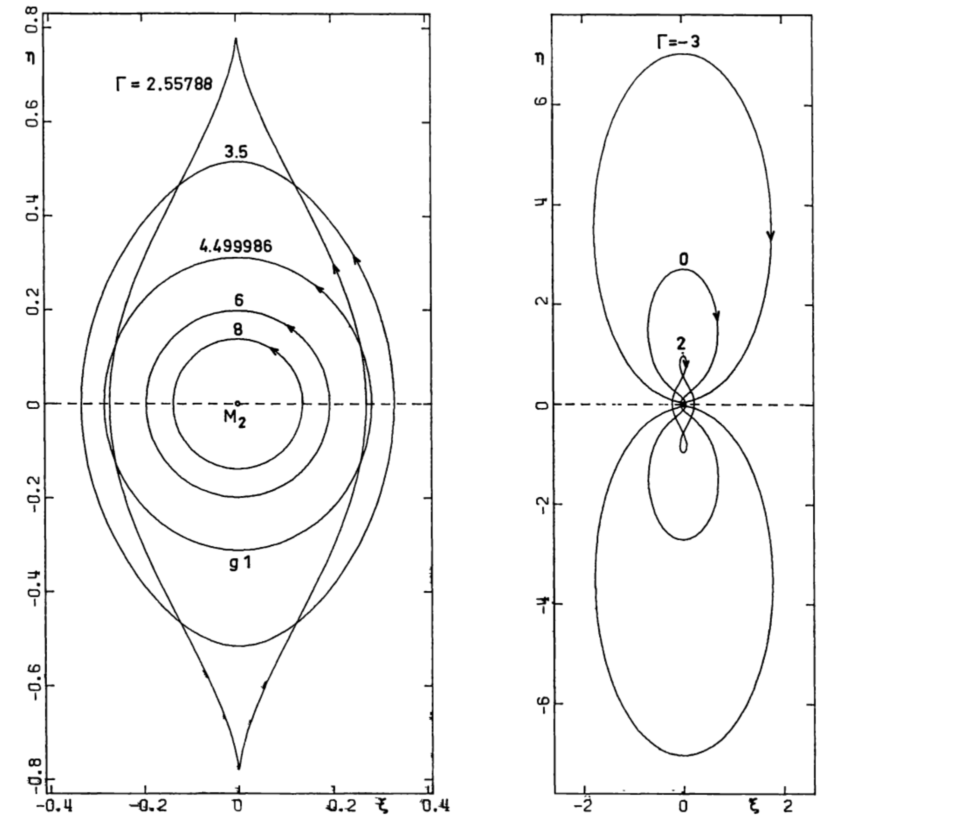
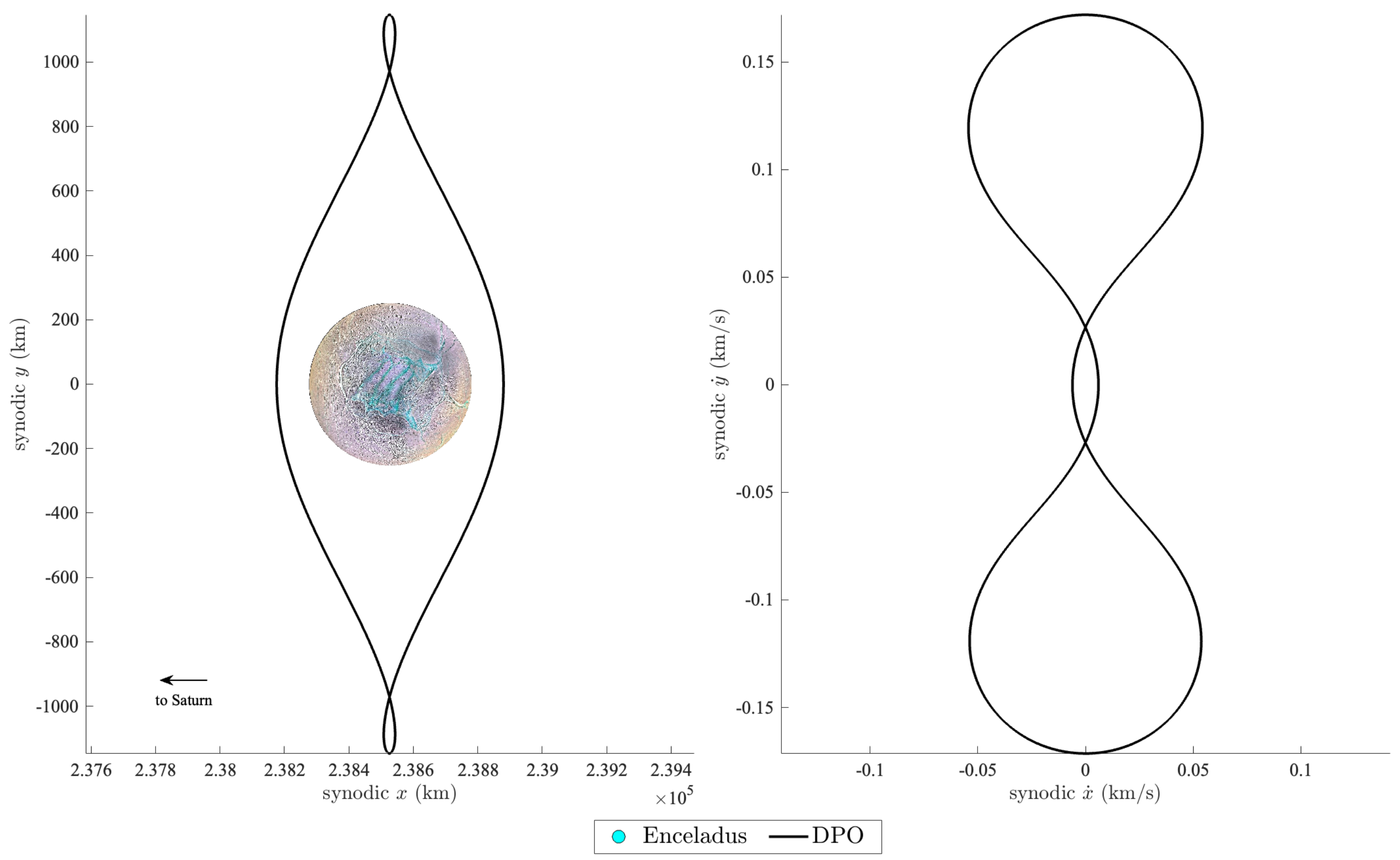
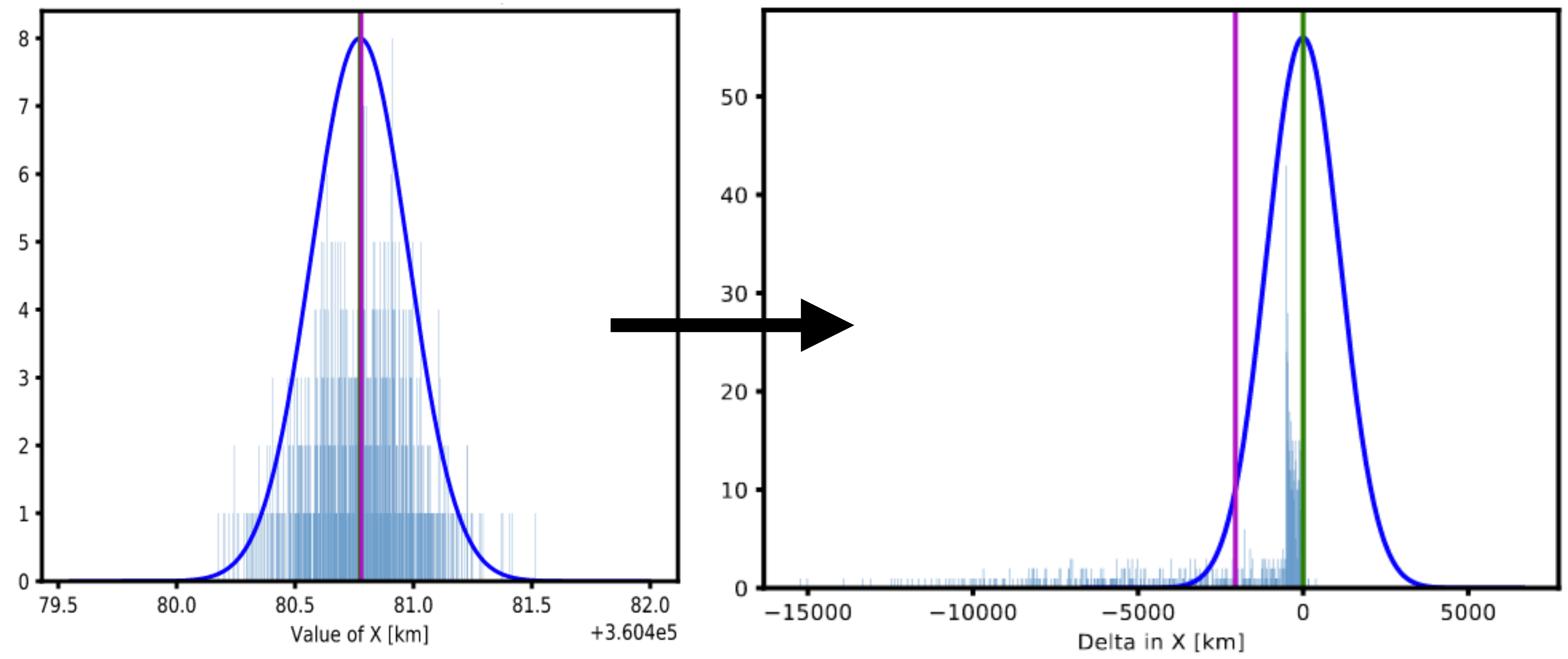
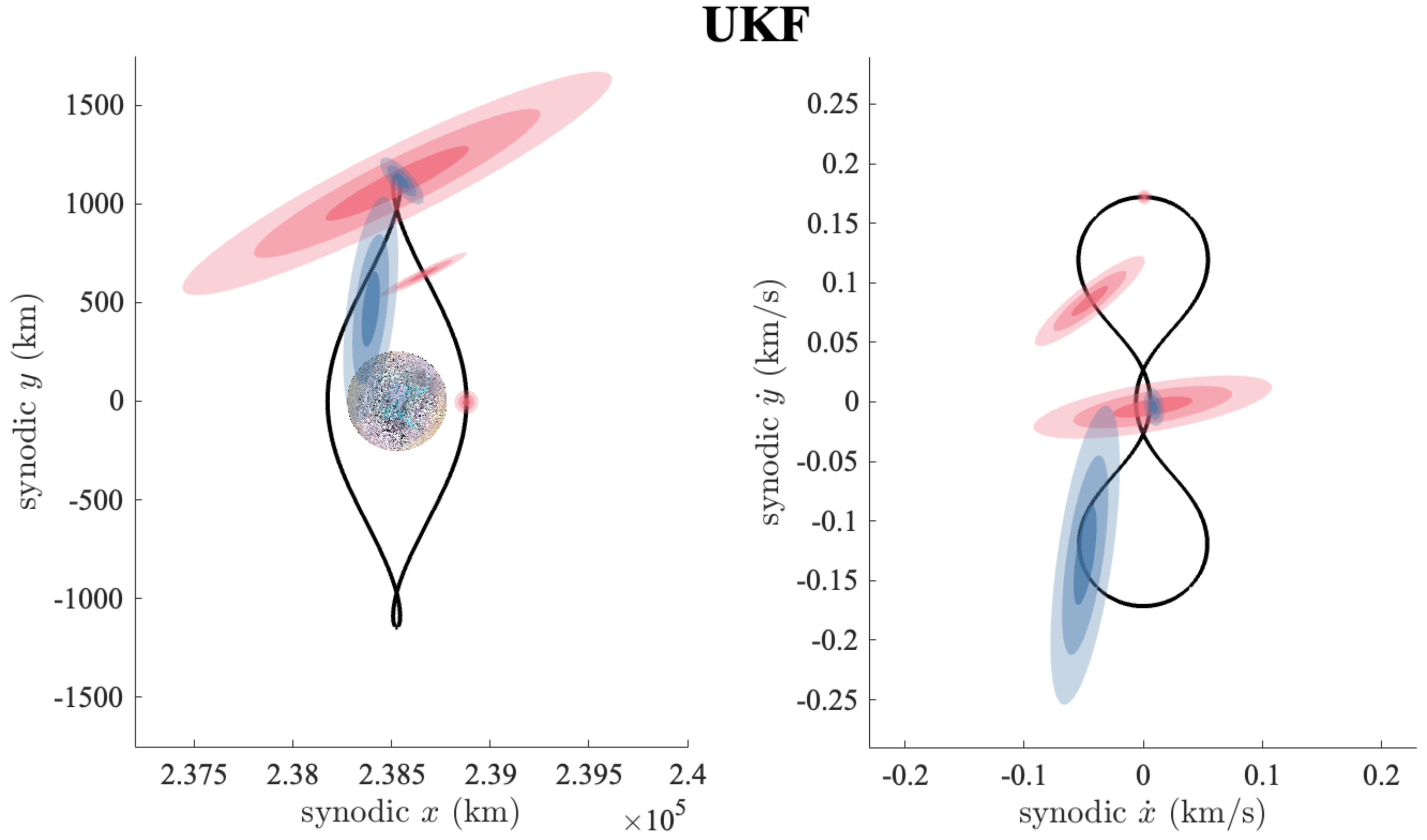


Fig. 4. Family g of periodic orbits. Note that the two parts of the figure have different scales. Orbits with $\Gamma > 4.499986$ are stable, the others are unstable



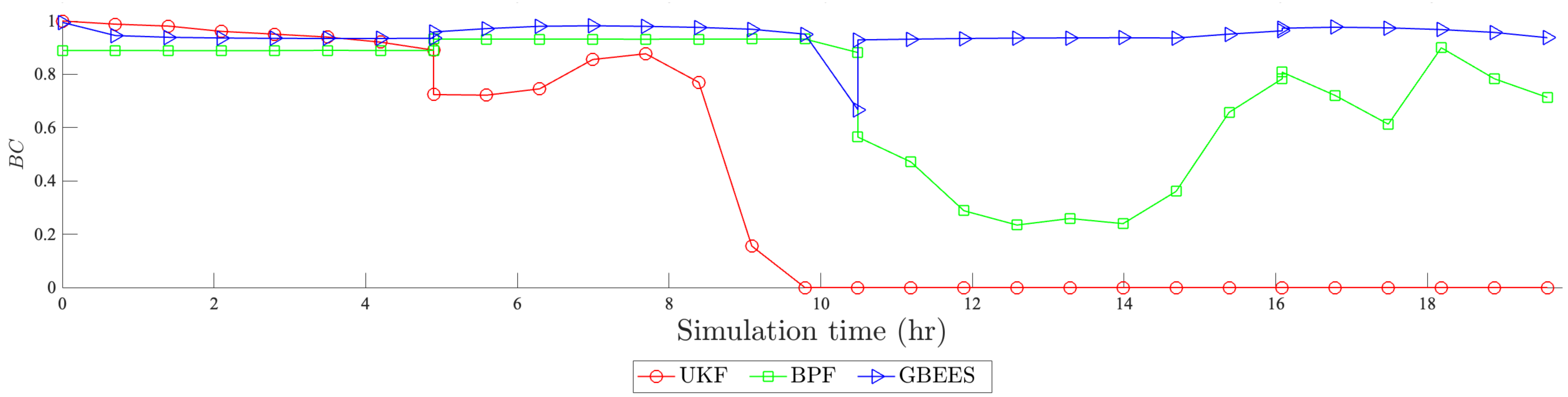
- For these trajectories, with nonlinear measurement updates, Gaussian filters tend to **diverge**



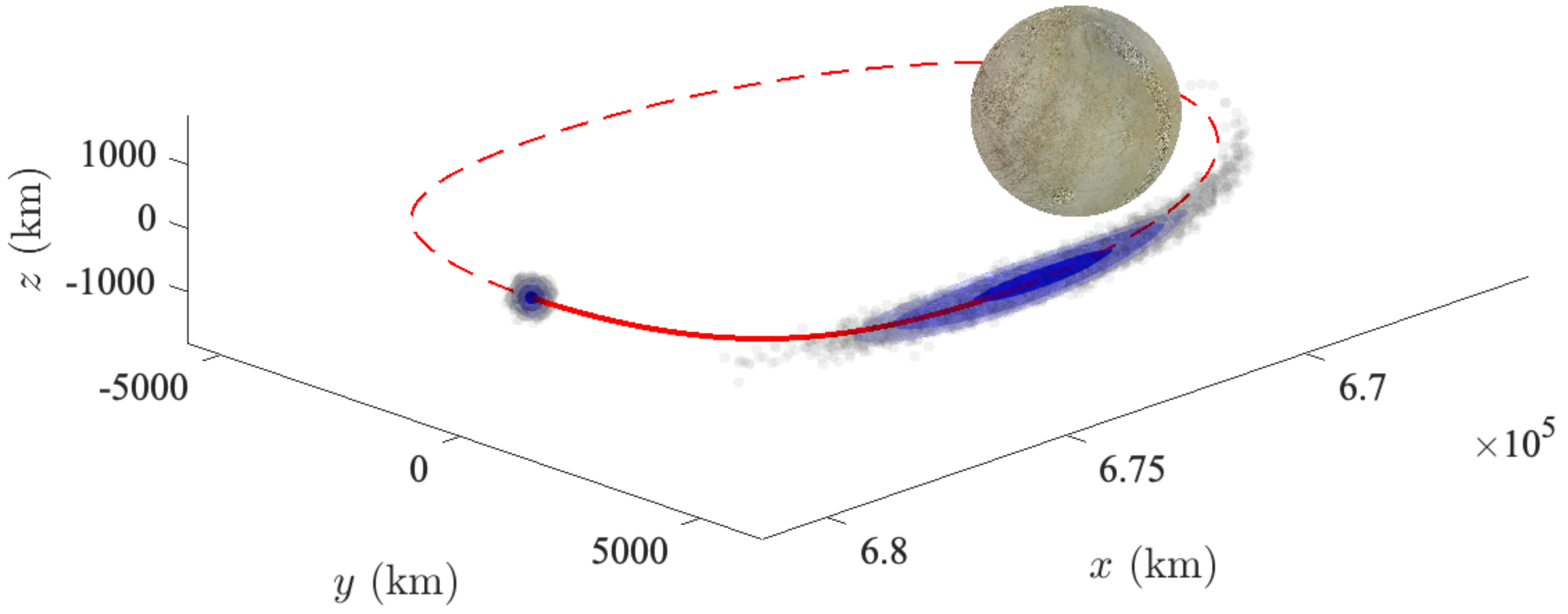


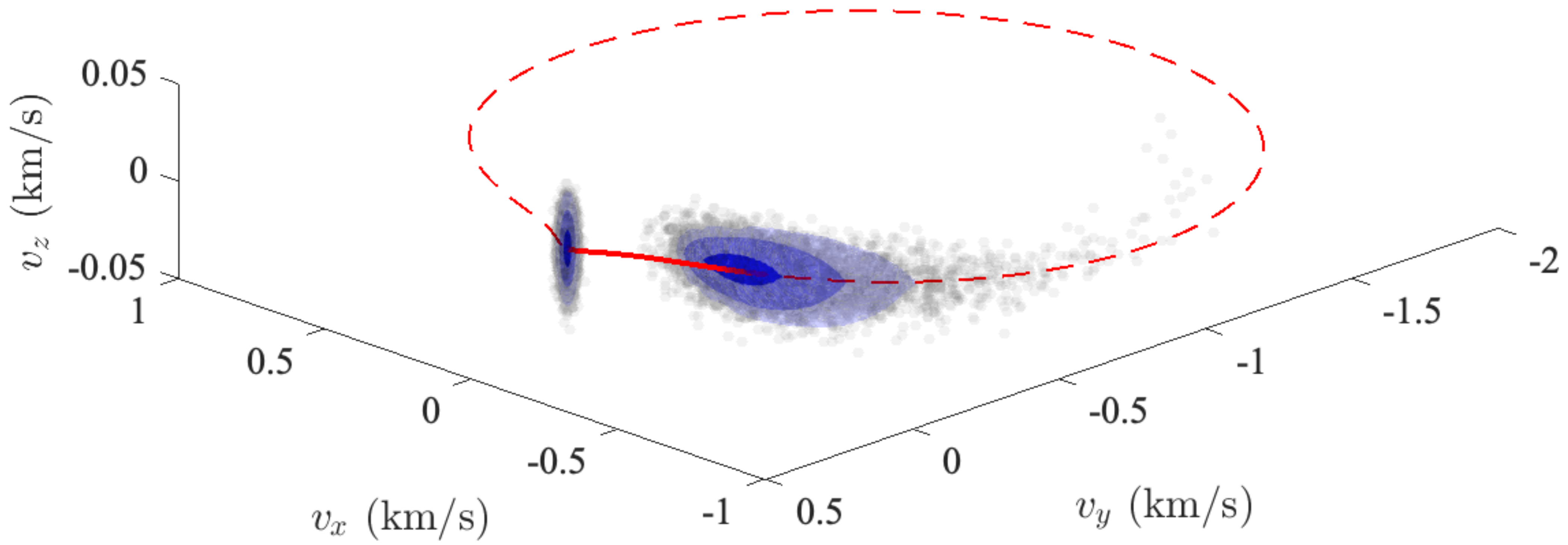
Conclusions

- GBEES proves to be an accurate, robust, and efficient alternative to the landscape of non-Gaussian RBFs



- **Up next:** higher-dimensional systems, parallelization, and ephemeris models (oh my!)





This investigation was supported by the NASA Space Technology Graduate Research Opportunities Fellowship (Grant #80NSSC23K1219)

Thanks to Dr. Ely and Dr. Lo for mentoring and co-mentoring me this summer, as well as for their invaluable insight and contributions.

GBEES can be found at: https://github.com/bhanson10/GBEES

Thank you for your time. Questions?