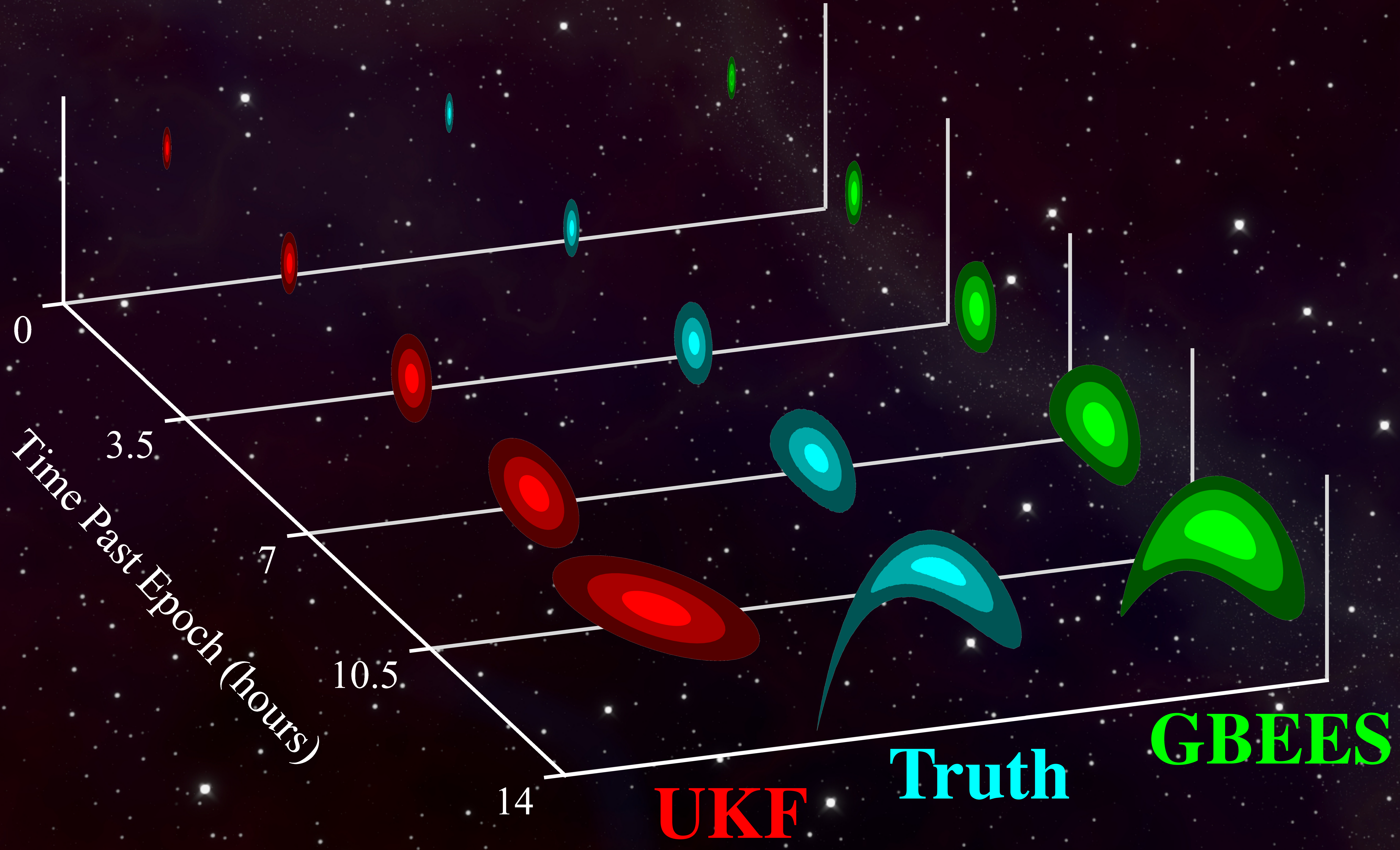
On the Validity of the Gaussian Assumption in the Jovian System: Evaluating Linear and Nonlinear Filters for Measurement-sparse Estimation

KePASSA 2024



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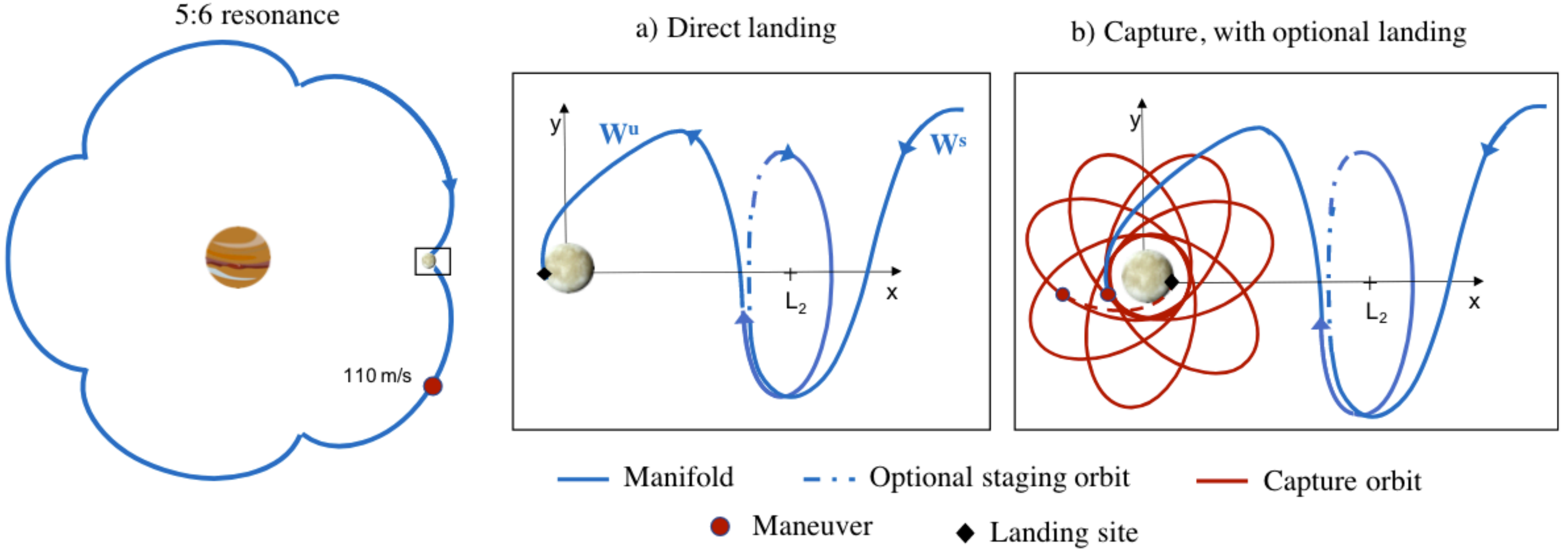
Case Study: Low-Energy Trajectories for Europa Lander

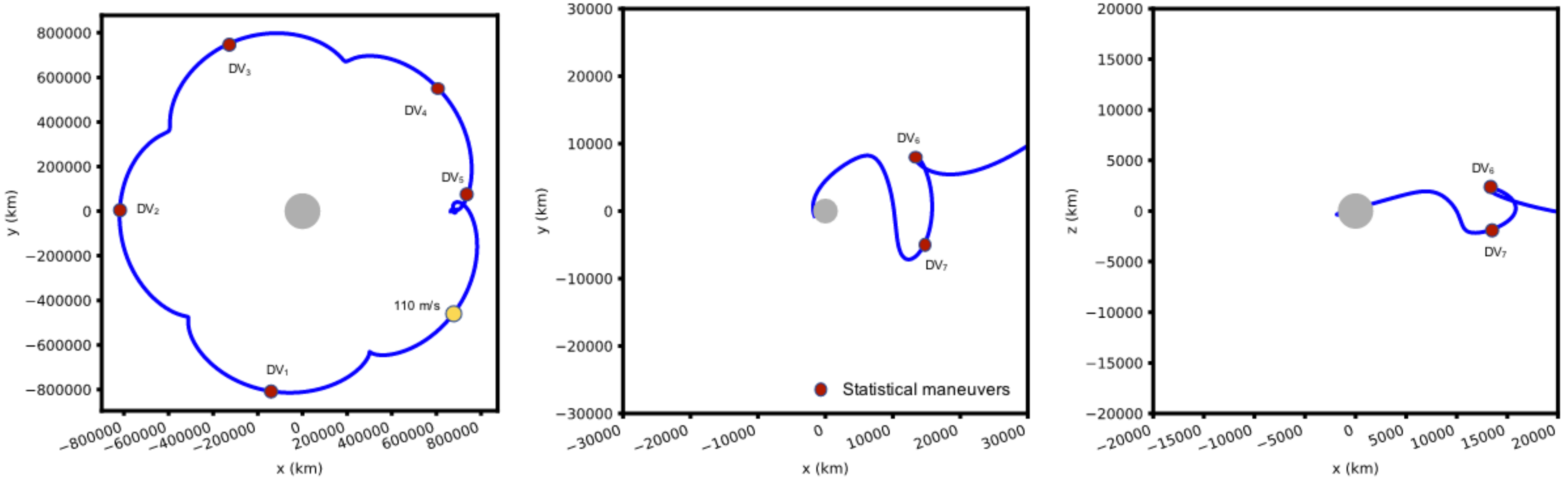
Time validity of the Gaussian assumption of uncertainty

- A theoretically ΔV-free, ballistic capture of a Europa lander realistically requires **statistical maneuvers** to maintain Gaussian error in position and velocity for linearized navigation techniques

Proposed ΔV-free ballistic capture



Actual trajectory with statistical maneuvers ΔV_i



McElrath, Tim, et al. "Navigating low-energy trajectories to land on the surface of Europa." NTRS. (2021)

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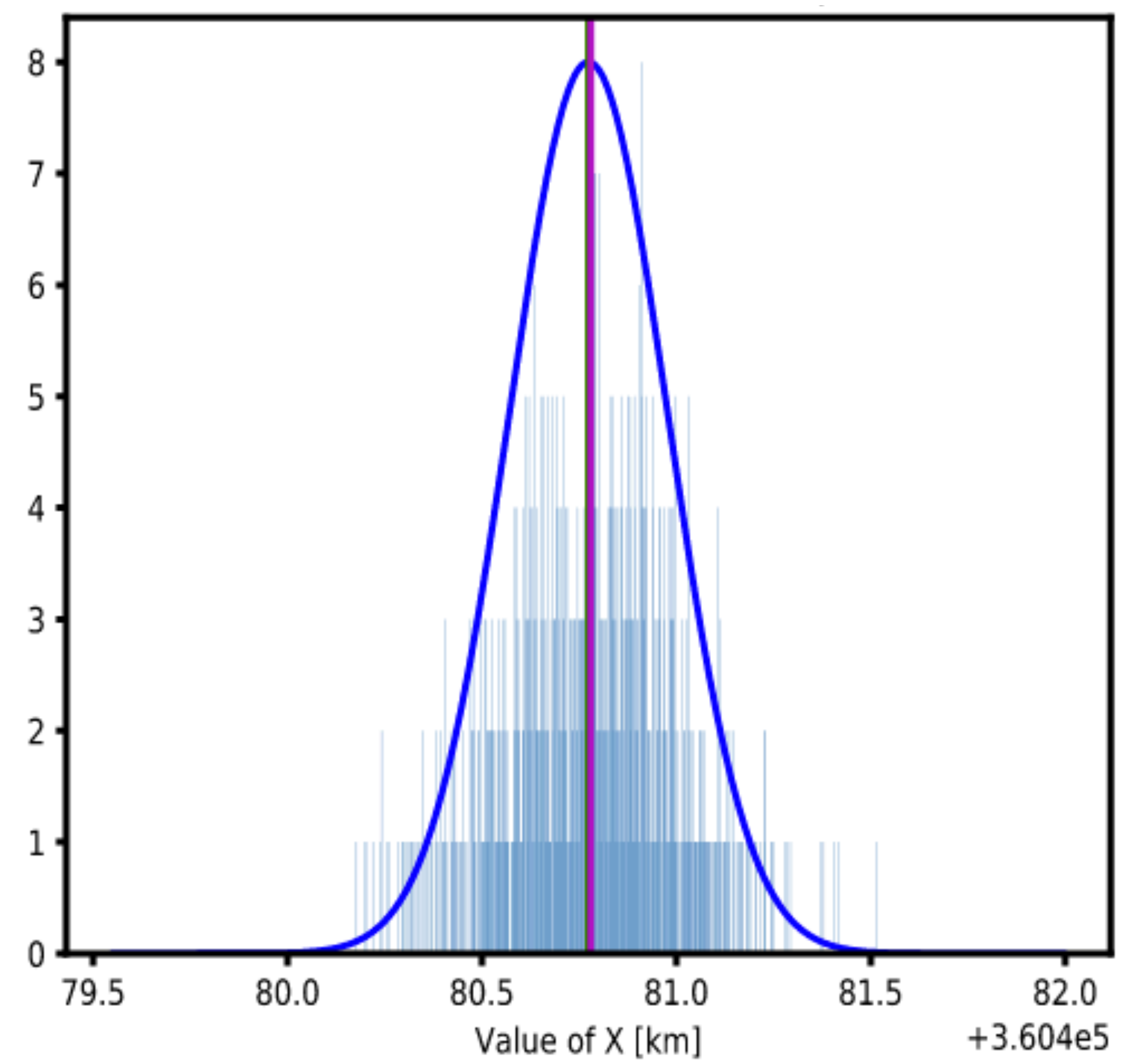
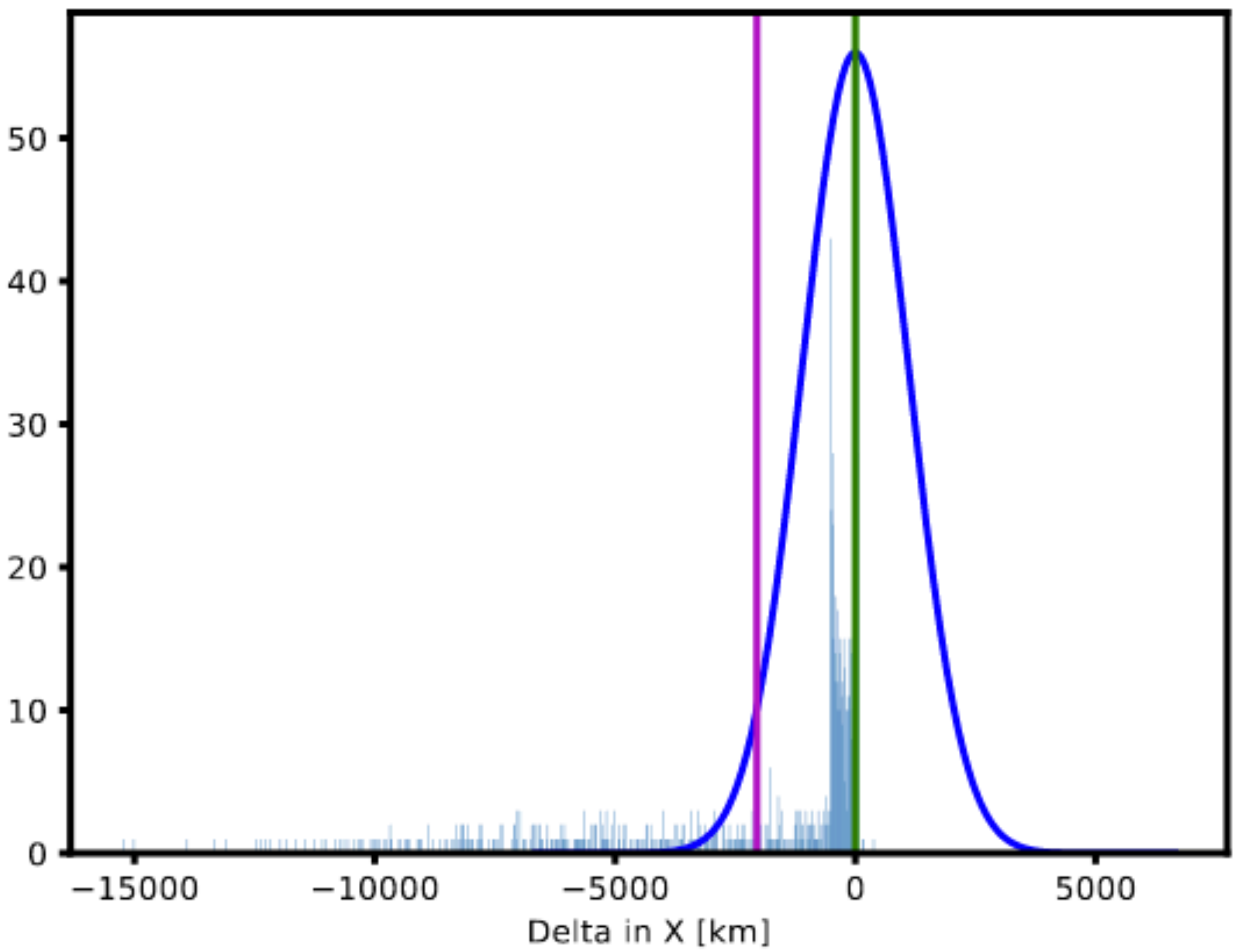
Proposed ΔV-free ballistic capture



Initial Gaussian uncertainty at leveraging maneuver



Final non-Gaussian uncertainty at Europa arrival

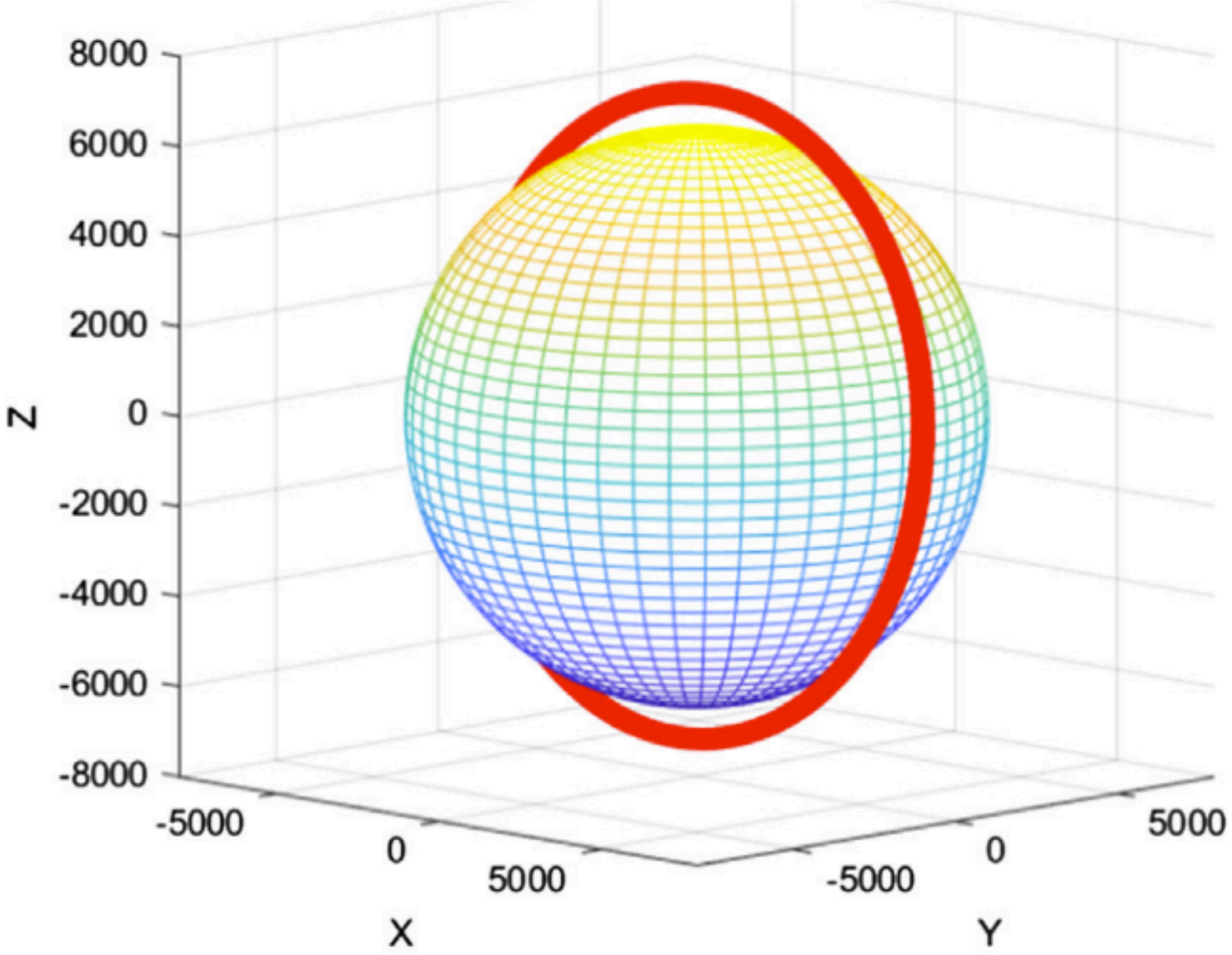
KEY QUESTION: What are the temporal limits of linear filters in the Jovian regime, and when might it be necessary to implement nonlinear filters?

McElrath, Tim, et al. "Navigating low-energy trajectories to land on the surface of Europa." NTRS. (2021)

Test Model: Low-Earth Orbit (LEO)

Review of measurement-sparse LEO estimation

- Previous work has focused on the efficacy of linear/nonlinear filters applied to LEO trajectories in measurement-sparse conditions
  - ✳ Initial condition resulting in highly-inclined, nearly-circular LEO
  - ✳ Propagated for 6 revolutions (4.94 hours) w/ RK8(7)
  - ✳ Negligible process noise ($Q = 0$)



$$\boldsymbol{x}_0 = \begin{bmatrix} a & (\mathrm{km}) \\ e & (\quad) \\ i & (\circ) \\ \Omega & (\circ) \\ \omega & (\circ) \\ M & (\circ) \end{bmatrix} = \begin{bmatrix} 7{,}078.0068 \\ 0.01 \\ 85^\circ \\ 0^\circ \\ 0^\circ \\ 0^\circ \end{bmatrix}$$

$$\sigma_r = 30 \ \mathrm{m}, \ \sigma_v = 0.3 \ \mathrm{m/s}$$

Dynamic Model	Description
Primary Body Gravity	70 x 70
Third-Body Perturbations	Sun and Moon
Atmospheric Drag	Cannonball
Solar Radiation Pressure	Cannonball

Yun, Sehyun, et al. "Kernel-based ensemble gaussian mixture filtering for orbit determination with sparse data." AdSpR. (2022)

Jovian Application: Framework Changes

Truth Model

- We plan to implement a similar framework as previously shown, applied to trajectories in the Jovian system, with the following important changes implemented:

I. Truth Model

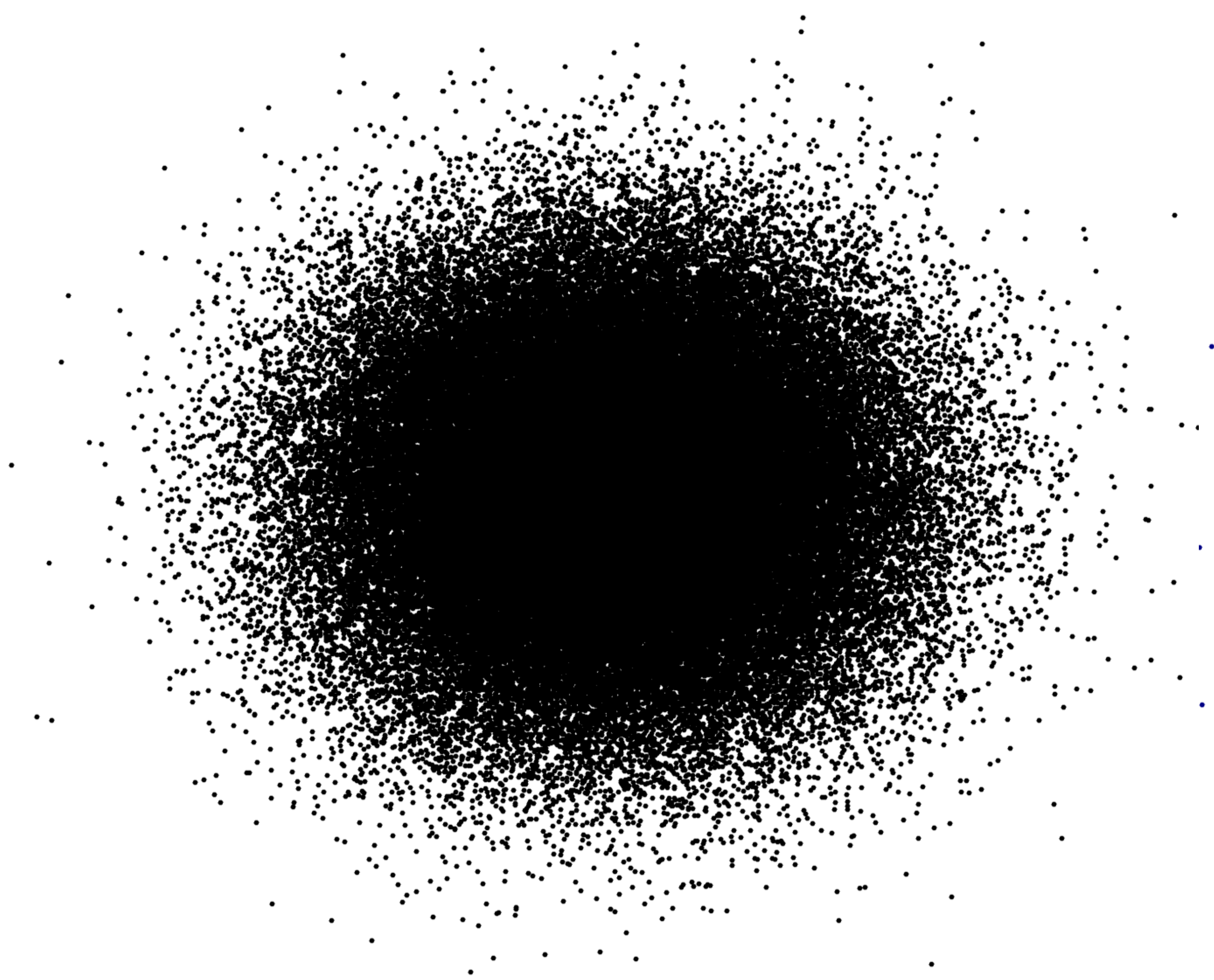
✳ A **high-resolution particle filter** will allow for **confidence interval** comparison with linear filters, providing more information than a high-resolution Monte Carlo distribution

For Monte Carlo/Particle Filter:

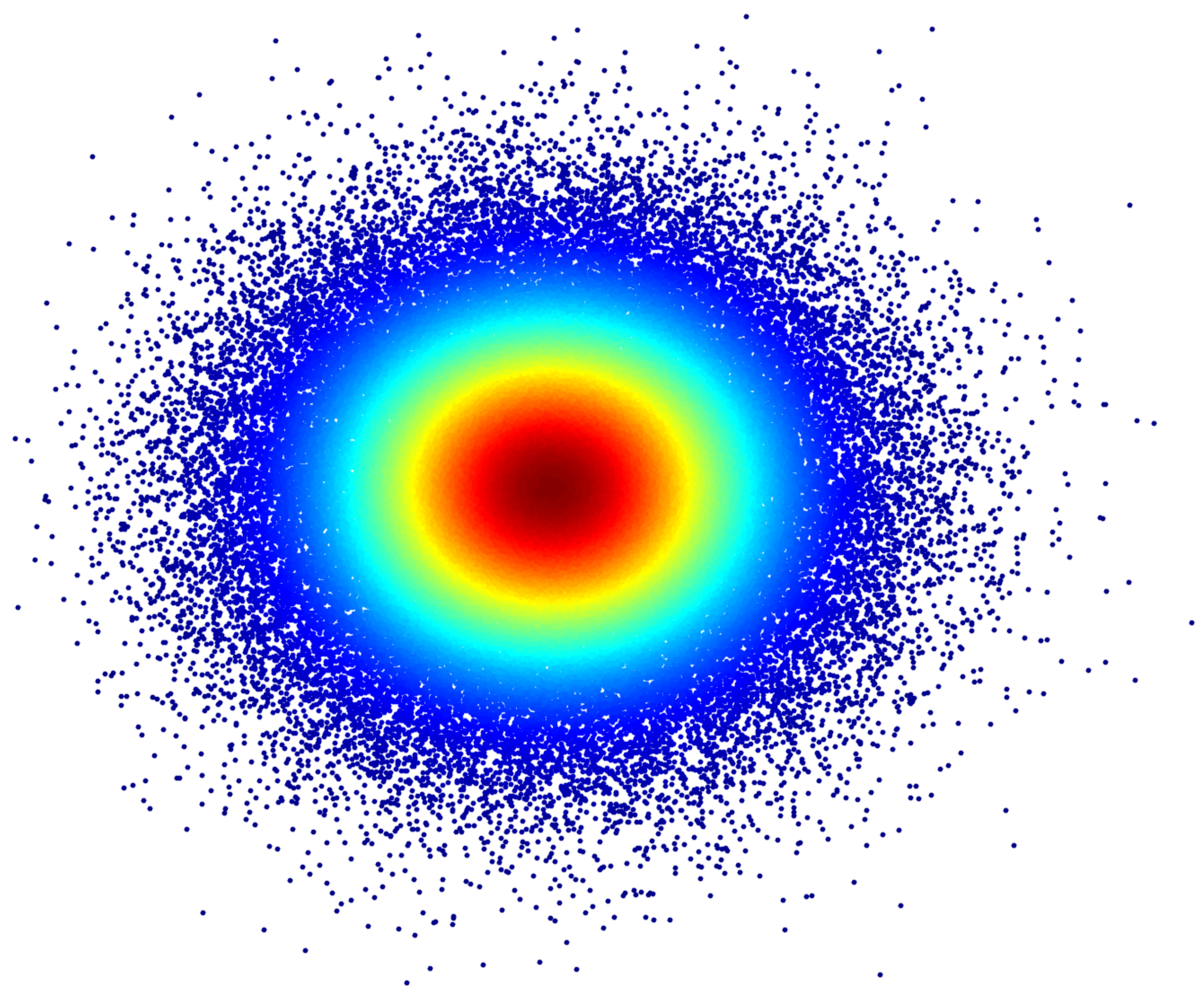
$$\{\boldsymbol{x}\} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$$

For Particle Filter only:

$$\{\boldsymbol{p}(\boldsymbol{x})\} \sim \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\}$$



Monte Carlo Interpretation



Particle Filter Interpretation

Jovian Application: Framework Changes

Distribution Comparison Metric

- We plan to implement a similar framework as previously shown, applied to trajectories in the Jovian system, with the following important changes implemented:

2. Distribution Comparison Metric

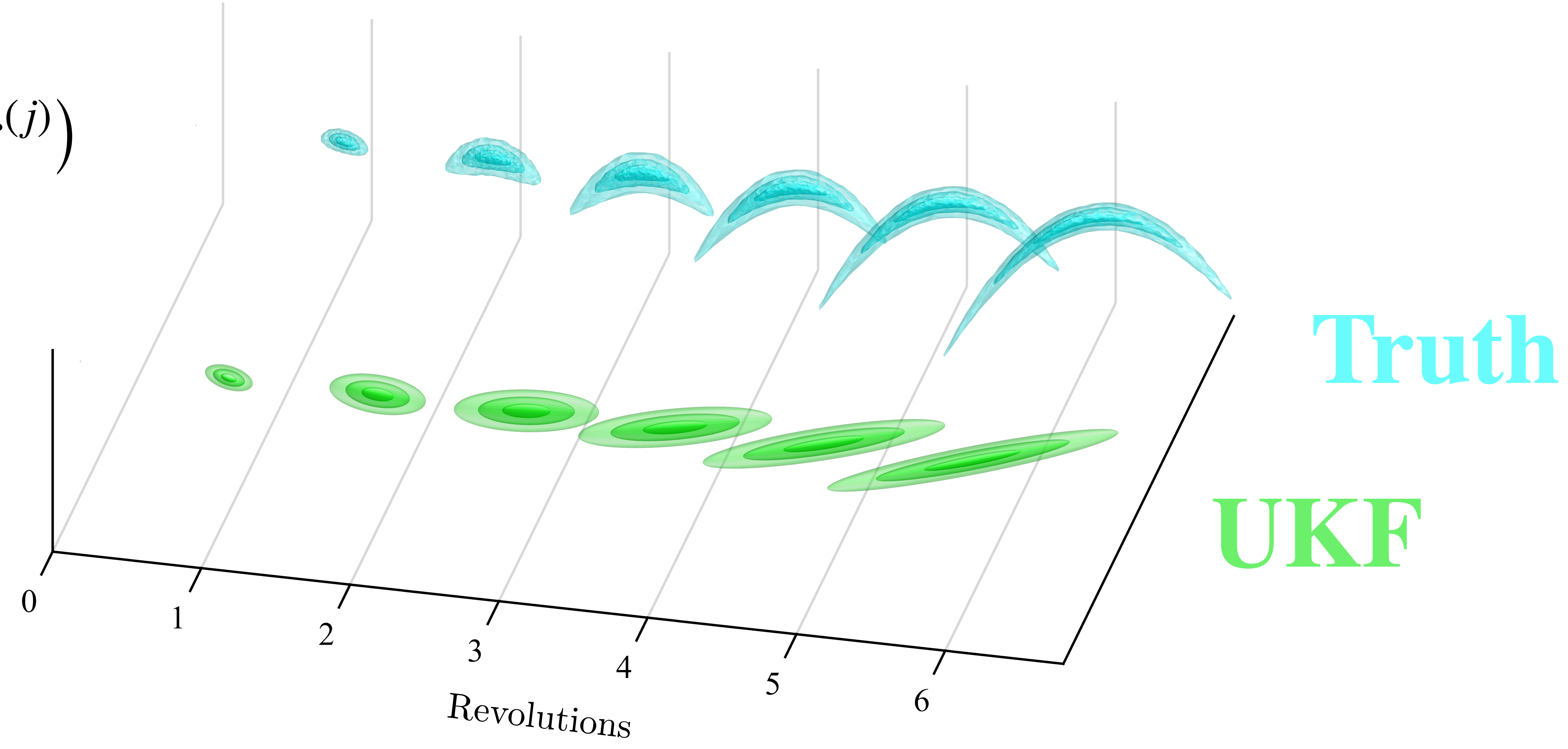
✴ The metric used to indicate diverge (previously SNEES) should consider **the true probability distribution is non-Gaussian** after enough propagation time without measurements

$$SNEES = \frac{1}{Md}\sum_{j=1}^{M}\left(\boldsymbol{x}^{(j)} - \hat{\boldsymbol{x}}^{(j)}\right)^T \left(\hat{\Sigma}^{(j)}\right)^{-1}\left(\boldsymbol{x}^{(j)} - \hat{\boldsymbol{x}}^{(j)}\right)$$

✦ **Problem:** Assumes Gaussian errors

$$D_{KL}(P||Q) = \sum_{x\in\chi} P(x)\,\log\left(\frac{P(x)}{Q(x)}\right)$$

✦ **Problem:** Diverges for extremely low probability events when distributions differ



Gaussian uncertainty propagated with Two-Body Dynamics becoming highly non-Gaussian

* $SNEES$: Scaled Normalized Estimation Error Squared

* D_{KL} : Kullback-Leibler Divergence

Jovian Application: Framework Changes

Propagation Conditions

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- We plan to implement a similar framework as previously shown, applied to trajectories in the Jovian system, with the following important changes implemented:

3. Propagation Conditions

✳ To test the limits of the linear filters, we plan on performing **"measurementless"** propagation

✦ We consider negligible process noise ($Q = 0$) and correct initial measurements ($\delta \boldsymbol{x}_0 = \boldsymbol{x}_0 - \hat{\boldsymbol{x}}_0 = 0$)

✳ Purely two-body dynamics will be propagated, so the following results are likely a **best-case scenario**

$$\boldsymbol{x} = \begin{bmatrix} a, & e, & i, & \Omega, & \omega, & M \end{bmatrix}^T, \quad \dot{\boldsymbol{x}} = \begin{bmatrix} 0, & 0, & 0, & 0, & 0, & \sqrt{\frac{\mu}{a^3}} \end{bmatrix}^T$$

✦ Future work will aim to feed the dynamics from an **ephemeris-level numerical propagator**

✳ Filter parameters:

Filter	Parameters
Particle Filter (truth)	Particles: 10^5
UKF	$\alpha = 10^{-3}$, $\beta = 2$, $\kappa = 0$
EnKF	Members: 10^4

Distribution Comparison Metric

Choosing a metric for Gaussian/non-Gaussian distribution comparison

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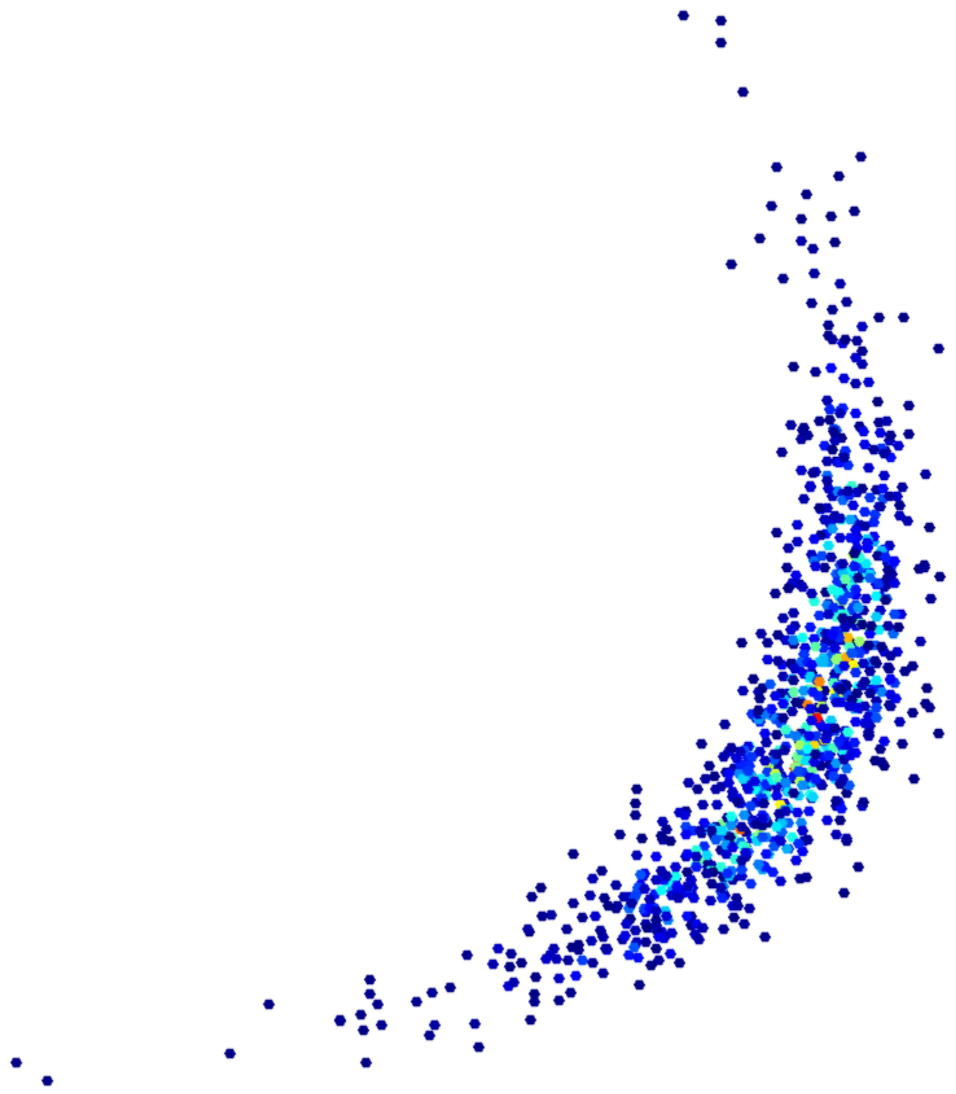
Point Mass Representation



	$p_1,$	$[x_1, x_2, \dots, x_d]_1$
	$p_2,$	$[x_1, x_2, \dots, x_d]_2$
	$p_3,$	$[x_1, x_2, \dots, x_d]_3$
	$p_4,$	$[x_1, x_2, \dots, x_d]_4$
	$p_5,$	$[x_1, x_2, \dots, x_d]_5$
	$p_n,$	$[x_1, x_2, \dots, x_d]_n$

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	$p_2,$	$[x_1, x_2, \dots, x_d]_2$
	$p_3,$	$[x_1, x_2, \dots, x_d]_3$
	$p_4,$	$[x_1, x_2, \dots, x_d]_4$
	$p_5,$	$[x_1, x_2, \dots, x_d]_5$
$\vdots$	$\vdots$	
	$p_n,$	$[x_1, x_2, \dots, x_d]_n$

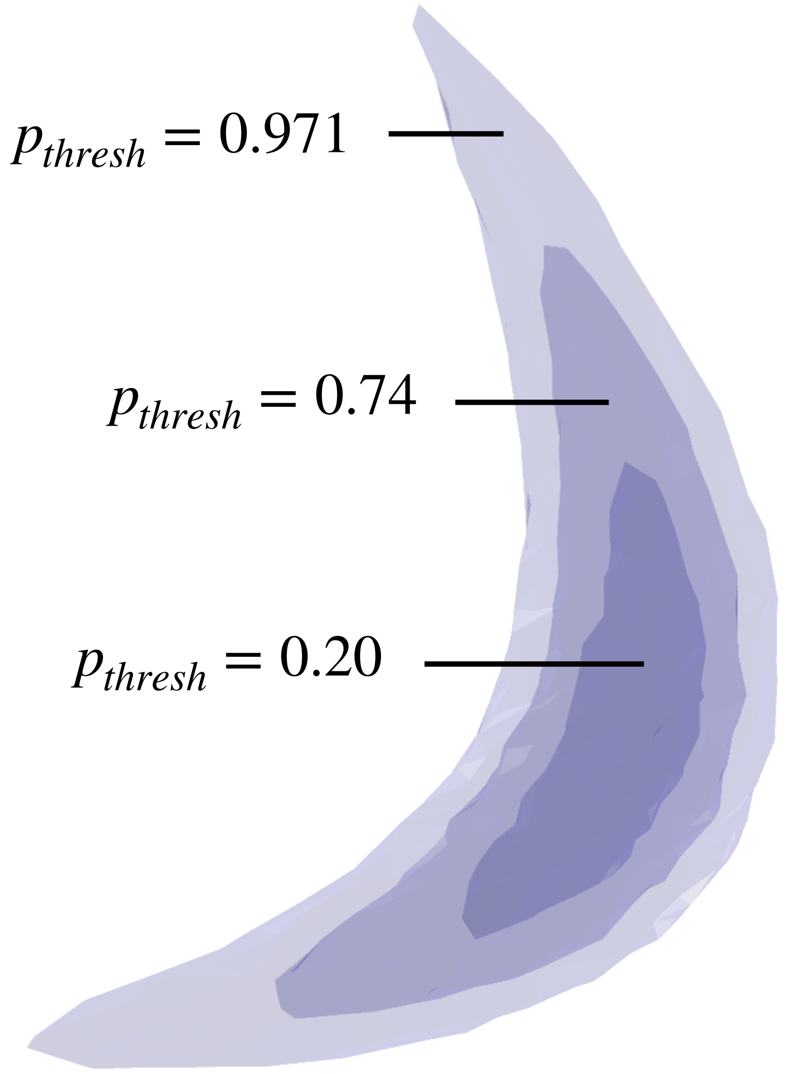
$$\{\boldsymbol{x}\}_{p^*}$$

where

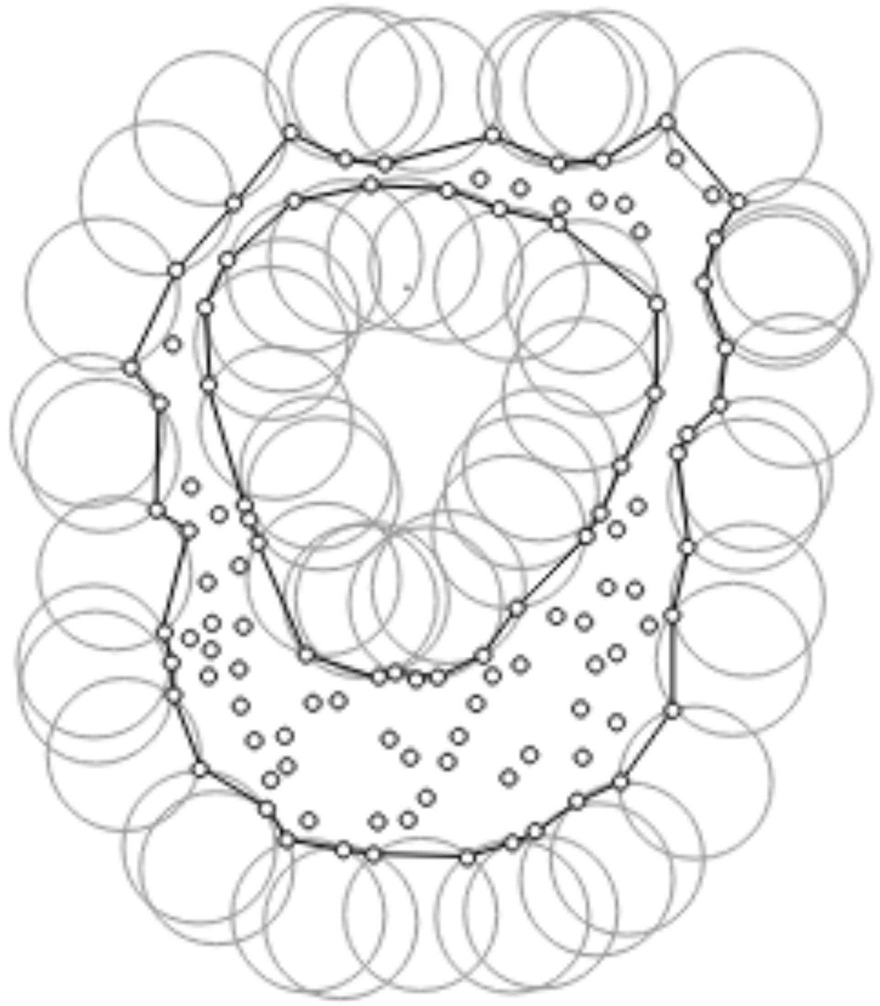
$$p^* = \sum_{i=1}^{M} p_i \leq p_{thresh}$$

	1σ	2σ	3σ
1D	68%	95%	99.7%
2D	39%	86%	98.9%
3D	20%	74%	97.1%

3D Isosurface Representation



α-Convex Hull Generation



Edelsbrunner, Herbert, et al. "Three-dimensional alpha shapes." ACM. (1994)

Distribution Comparison Metric

Choosing a metric for Gaussian/non-Gaussian distribution comparison

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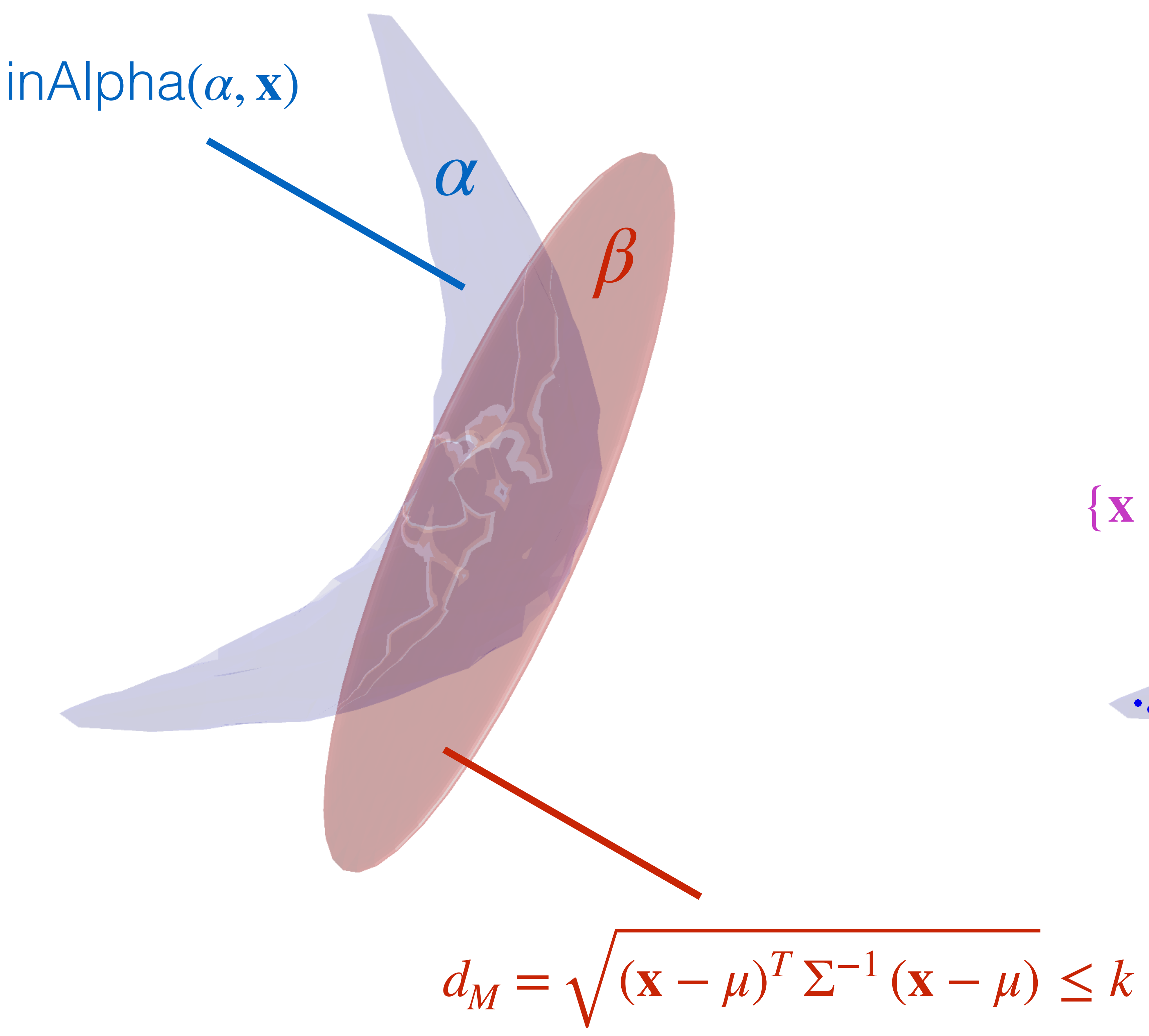
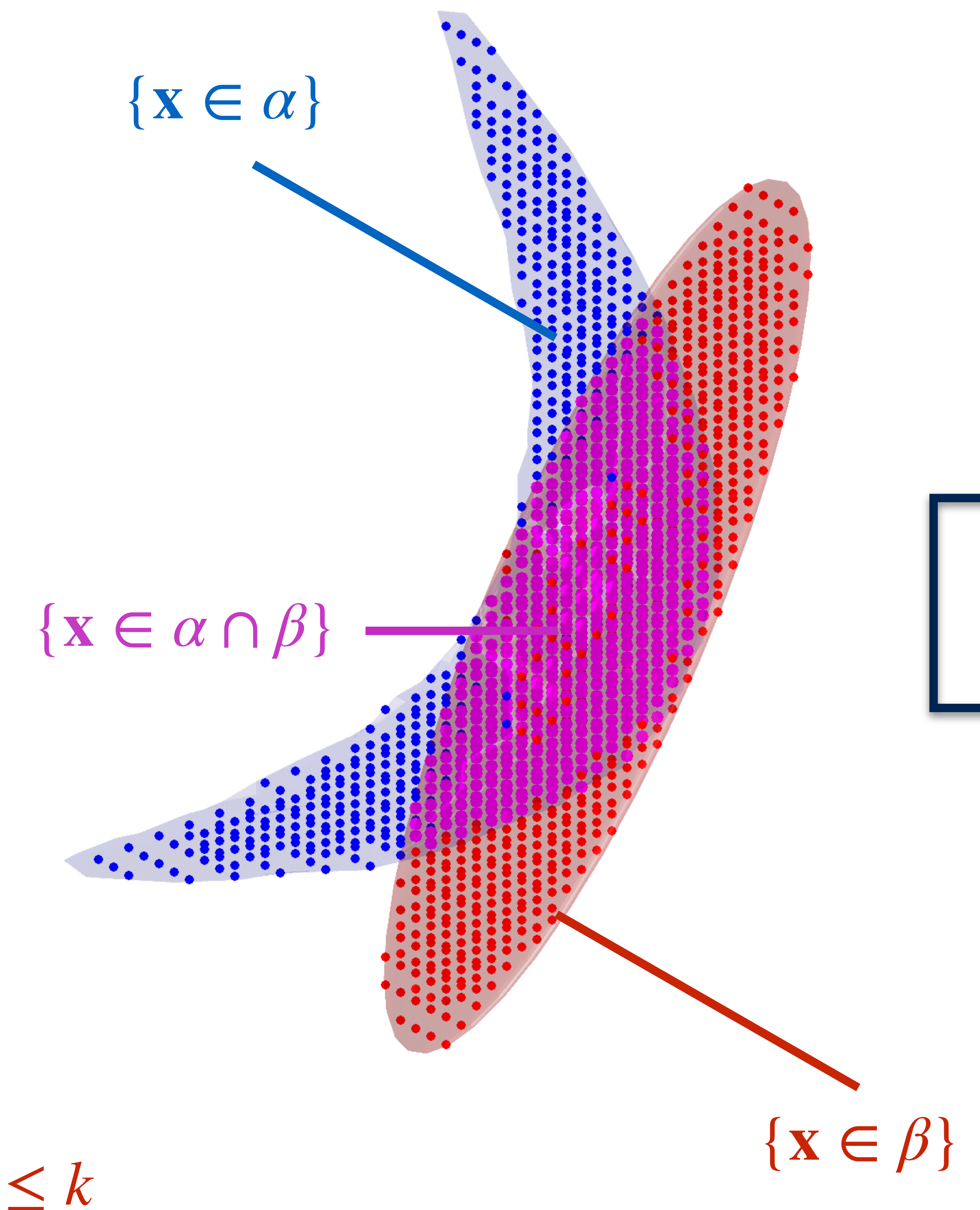
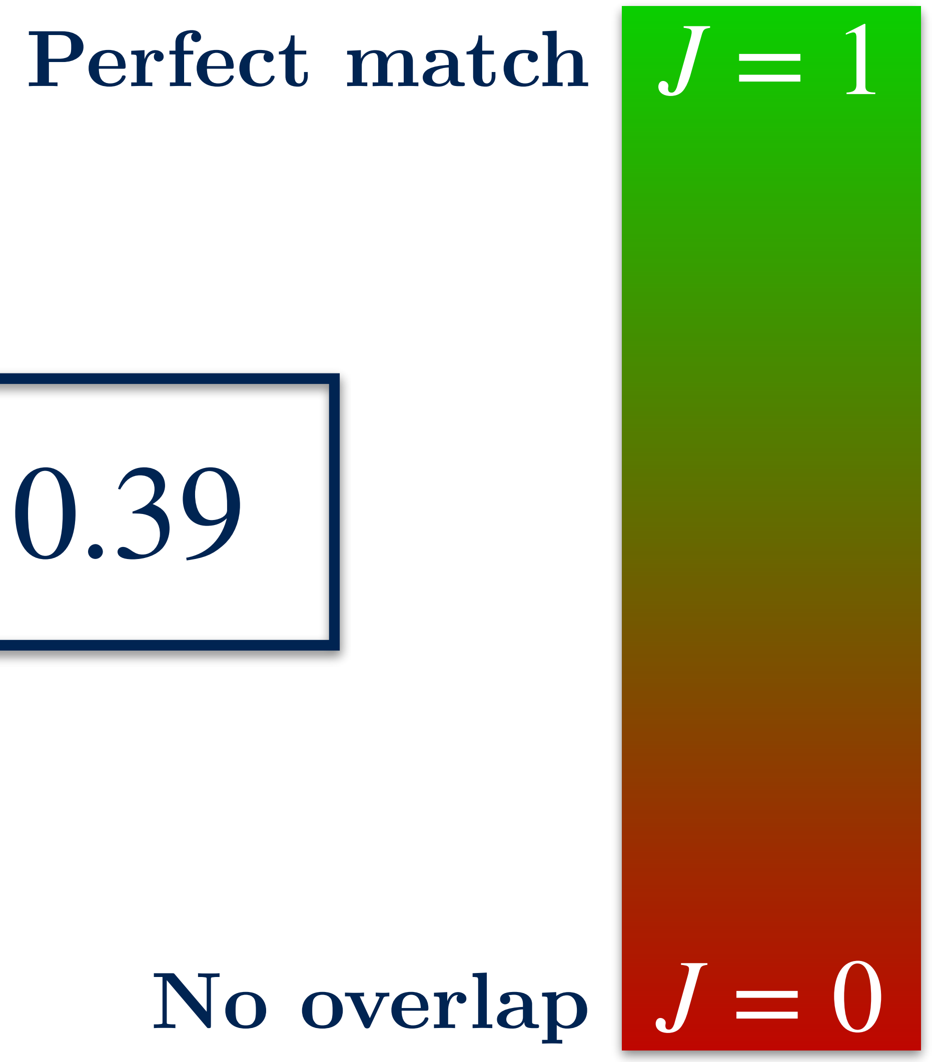
Distributions of interest



Discretization



$$J(\alpha, \beta) \approx 0.39$$



Jaccard Index $\equiv J(\alpha,\beta) = \dfrac{|\{x \in \alpha \cap \beta\}|}{|\{x \in \alpha \cup \beta\}|} = \dfrac{|\{x \in \alpha \cap \beta\}|}{|\{x \in \alpha\}| + |\{x \in \beta\}| - |\{x \in \alpha \cap \beta\}|}$ where $|\cdot| =$ size of set, $x \in \mathbb{R}^3$

Jovian Application: Low-Europa Orbit

Revised framework applied to measurement-sparse Jovian estimation

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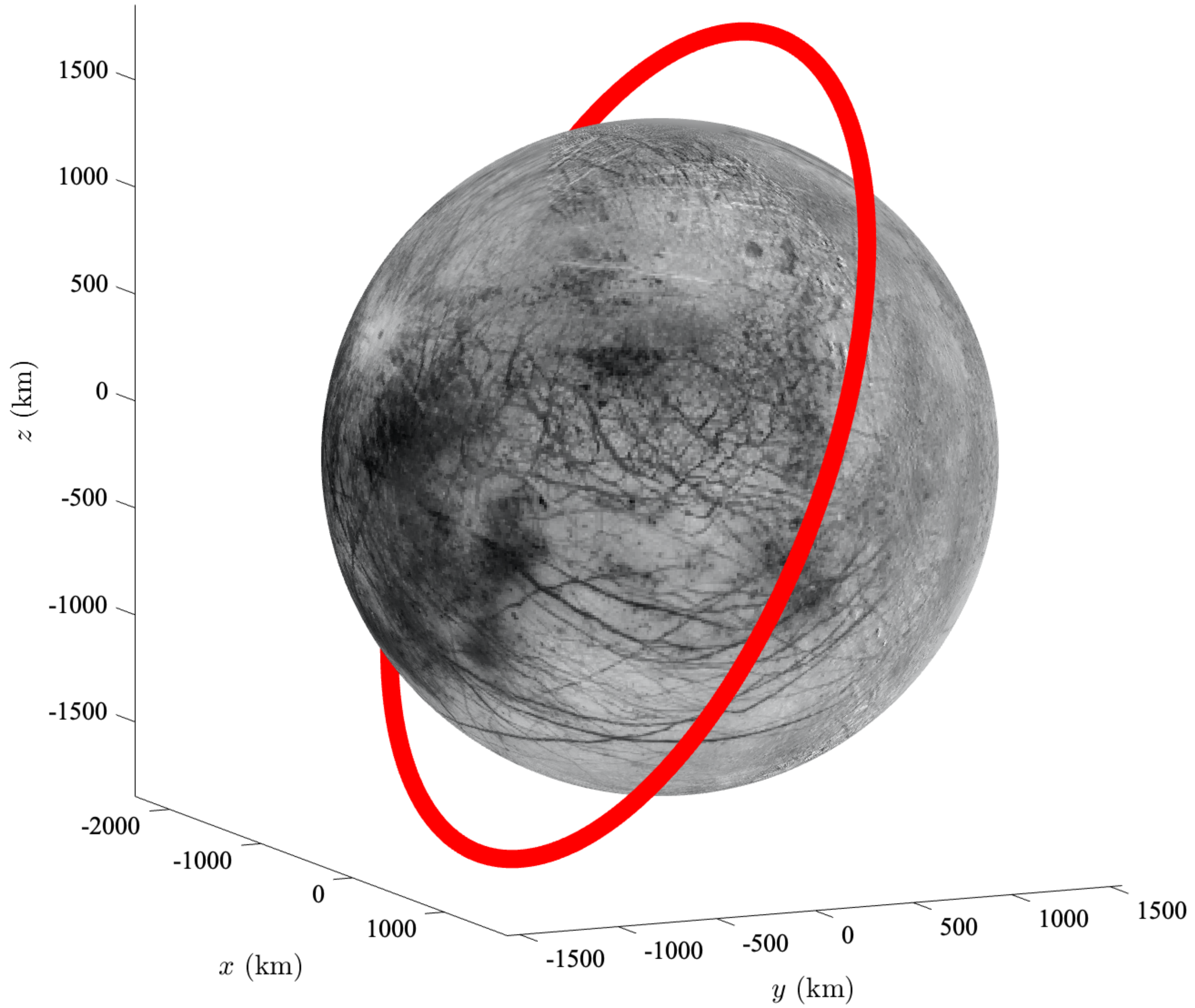
- Implement linear filter estimation with new comparison framework on Jovian trajectory:
  - ✶ Initial condition resulting in highly-inclined, low-Europa orbit
  - ✶ Propagated for 4 revolutions (11.279 hours) w/ RK8(7)
  - ✶ No measurements and negligible process noise
  - ✶ α-convex hull comparison metric



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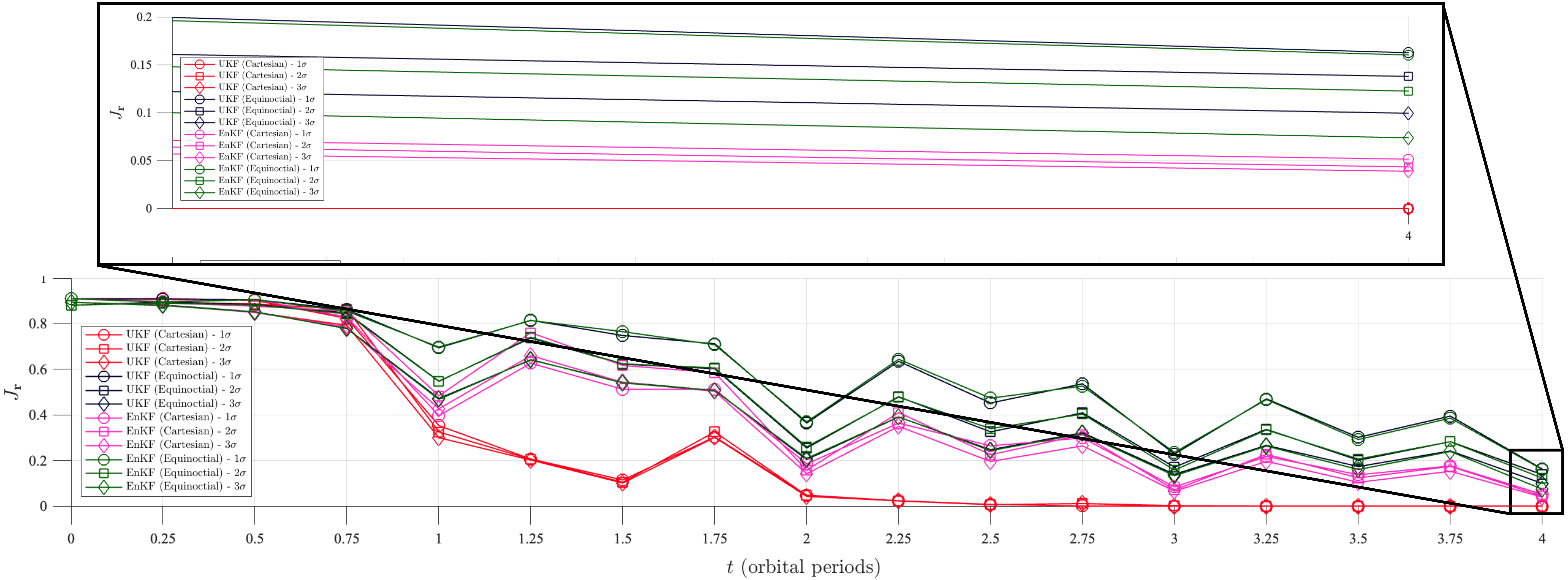
$$\sigma_r = 1 \text{ km}, \ \sigma_v = 1 \text{ m/s}$$

Filter	Parameters
Particle Filter (truth)	Particles: 10^5
UKF	$\alpha = 10^{-3}$, $\beta = 2$, $\kappa = 0$
EnKF	Members: 10^4

Lara, Martin, et al. "On the design of a science orbit about Europa." JPL. (2006)
Schenk, Paul, et al. "A very young age for true polar wander on Europa from related fracturing." GeoRL. (2020)

Jovian Application: Low-Europa Orbit

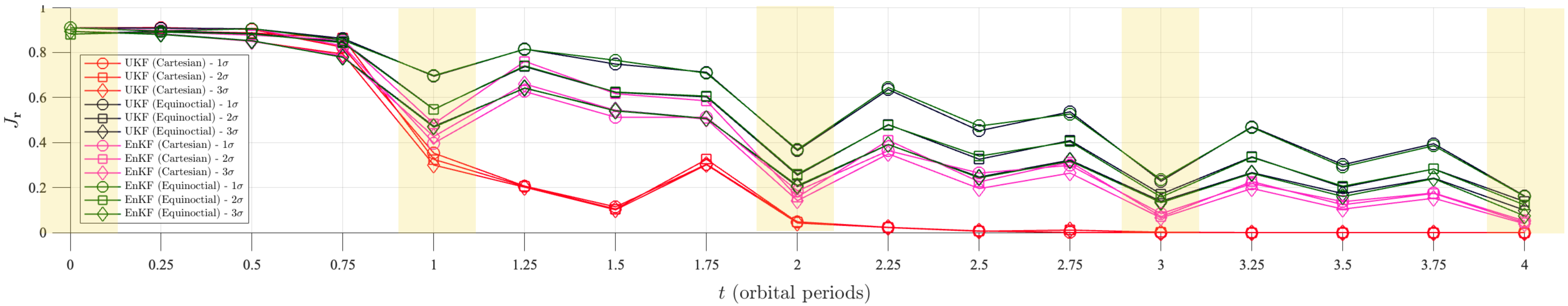
Evaluating the efficacy of linear filters for measurement-sparse estimation



J_r	UKF (Cartesian)	UKF (Equinoctial)	EnKF (Cartesian)	EnKF (Equinoctial)
1σ	N/A	**0.1763**	0.0445	0.1727
2σ	N/A	**0.1414**	0.0427	0.1237
3σ	N/A	**0.1049**	0.0366	0.0800

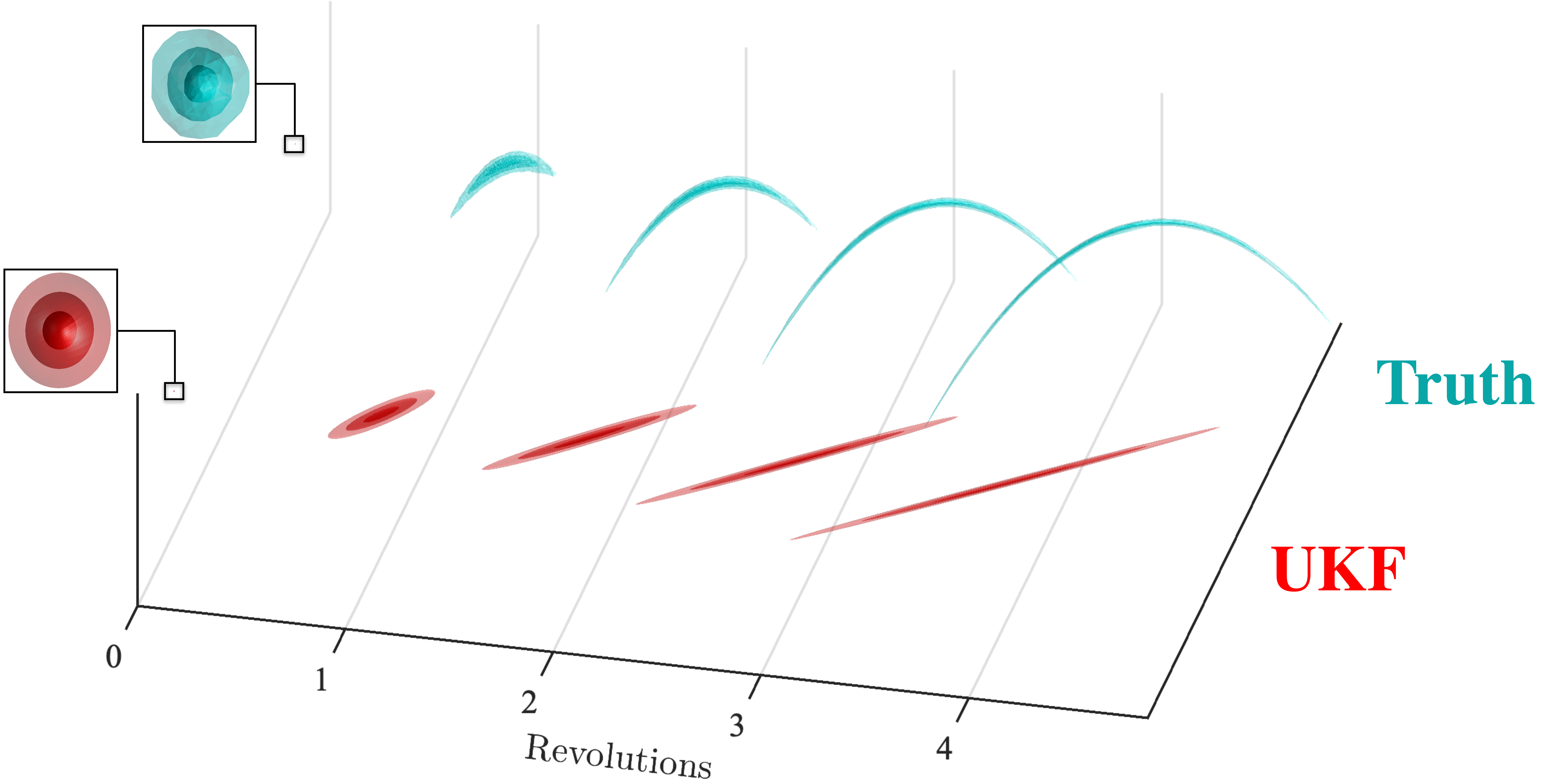
Jovian Application: Low-Europa Orbit

Evaluating the efficacy of linear filters for measurement-sparse estimation



Jovian Application: Low-Europa Orbit

Evaluating the efficacy of linear filters for measurement-sparse estimation



Motivation for New Nonlinear Filters

Addressing the shortcomings of the particle filter

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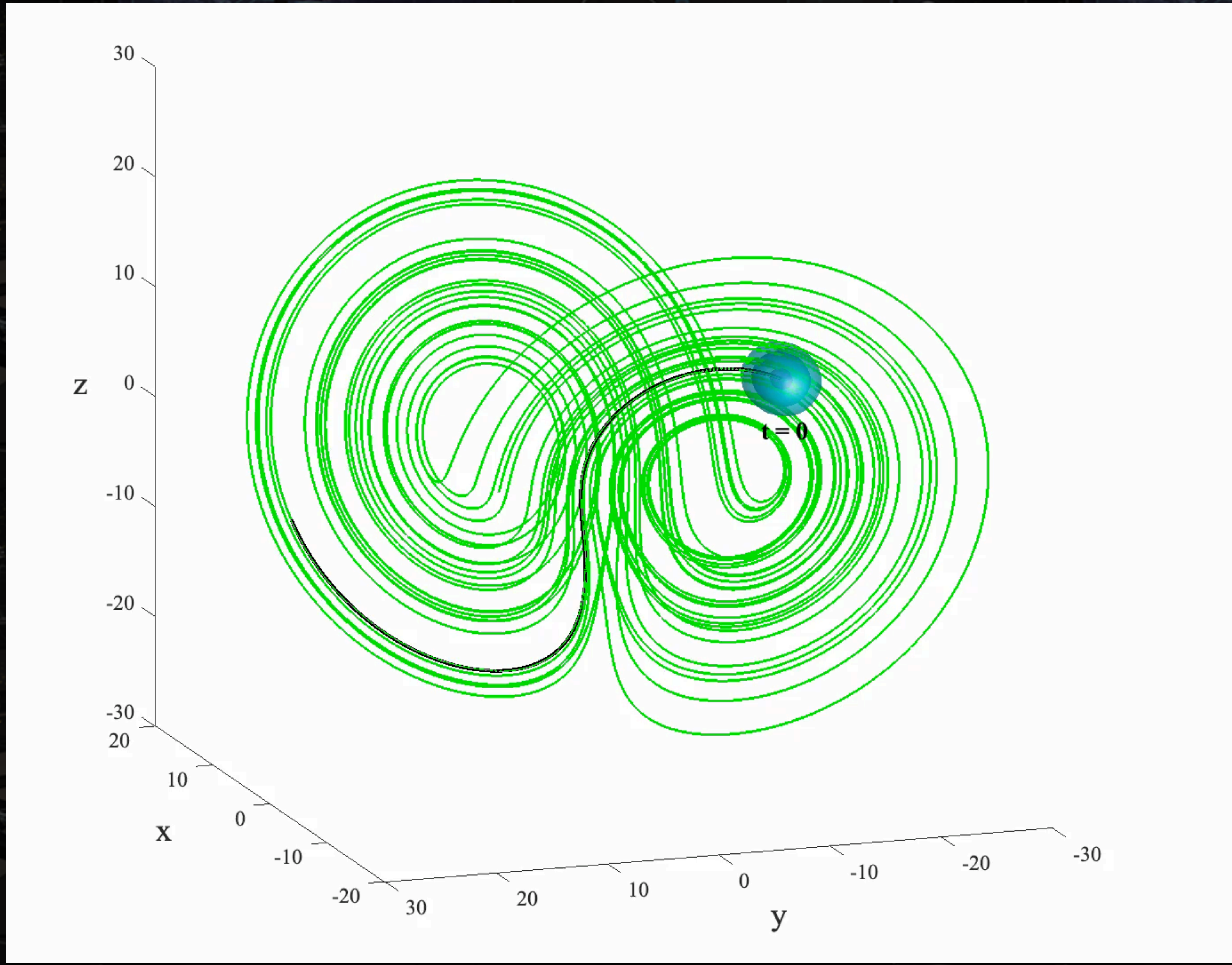
- To address the shortcomings of the linear filter, we utilize...



Grid-based Bayesian Estimation Exploiting Sparsity (GBEES)

An efficient Bayesian estimation method for representing and propagating uncertainty

GBEES is a 2nd-order accurate, Godunov finite volume method that treats probability as a fluid, flowing the PDF through phase space subject to the dynamics of the system. Because of its formulation, it can handle deterministic/stochastic systems while maintaining resolution.

Nonlinear Filter Comparison

Grid-based Bayesian Estimation Exploiting Sparsity (GBEES)

- GBEES consists of two distinct processes, one performed in **continuous-time**, the other in **discrete-time**:

1. The probability distribution function $p_{\mathbf{x}}(\mathbf{x}', t)$ is continuous-time marched via the **Fokker-Planck Equation**:

$$\frac{\partial p_{\mathbf{x}}(\mathbf{x}', t)}{\partial t} = -\frac{\partial f_i(\mathbf{x}', t) p_{\mathbf{x}}(\mathbf{x}', t)}{\partial x_i'} + \frac{1}{2}\frac{\partial^2 q_{ij} p_{\mathbf{x}}(\mathbf{x}', t)}{\partial x_i' \partial x_j'}$$

   - ✳ f_i: advection (EOMs) in the i^{th} dimension
   - ✳ q_{ij}: $(i, j)^{\text{th}}$ element of the spectral density ($Q = 0$, PDE is hyperbolic)

2. At discrete-time interval t_k, measurement y_k updates $p_{\mathbf{x}}(\mathbf{x}', t)$ via **Bayes' Theorem**:

$$p_{\mathbf{x}}(\mathbf{x}', t_{k+}) = \frac{p_{\mathbf{y}}(\mathbf{y}_k \,|\, \mathbf{x}') p_{\mathbf{x}}(\mathbf{x}', t_{k-})}{C}$$

   - ✳ $p_{\mathbf{x}}(\mathbf{x}', t_{k+})$: a posteriori distribution
   - ✳ $p_{\mathbf{y}}(\mathbf{y}_k \,|\, \mathbf{x}')$: measurement distribution
   - ✳ $p_{\mathbf{x}}(\mathbf{x}', t_{k-})$: a priori distribution
   - ✳ C: normalization constant

Nonlinear Filter Comparison

Grid-based Bayesian Estimation Exploiting Sparsity (GBEES)

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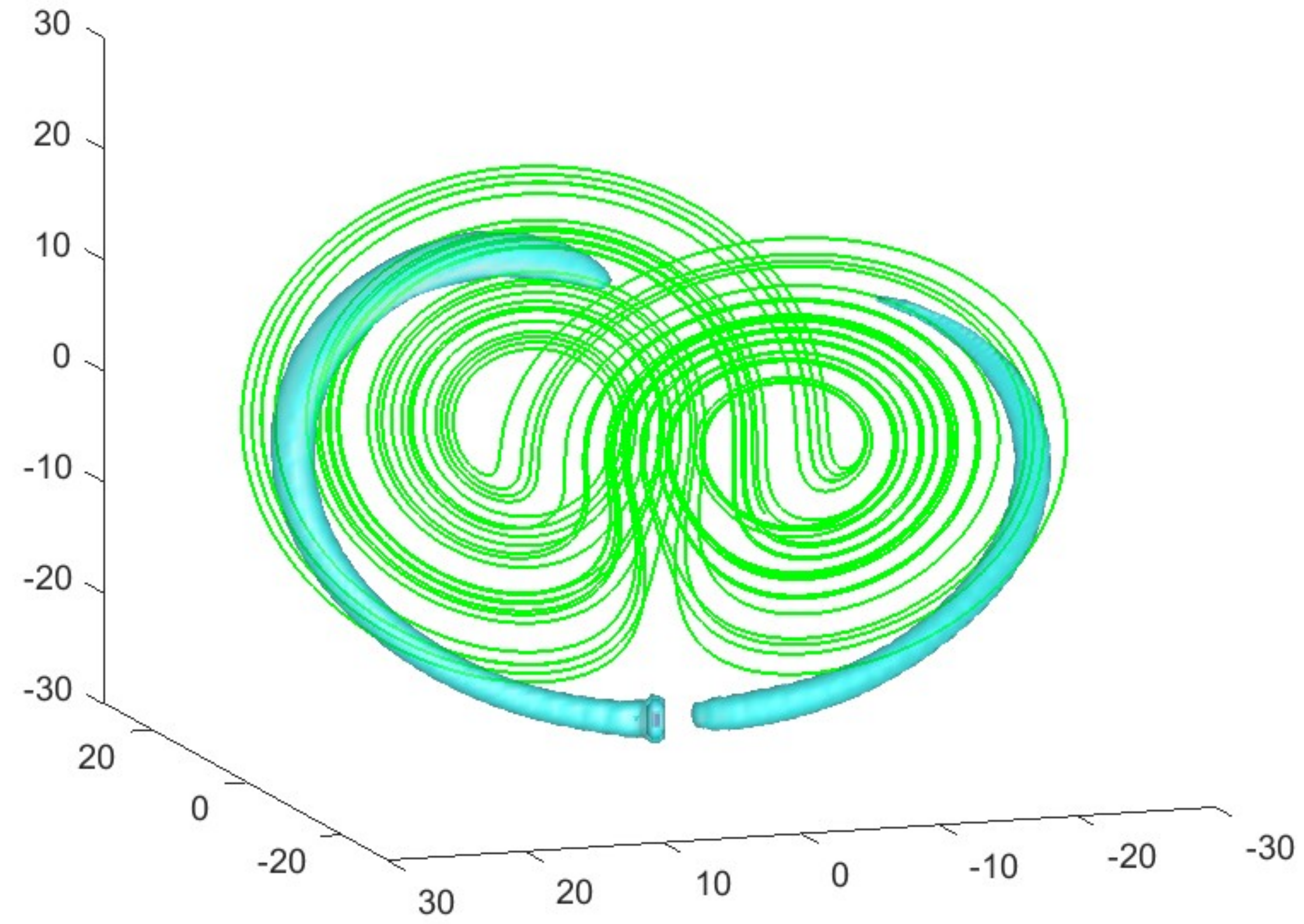

A priori

Nonlinear Filter Comparison

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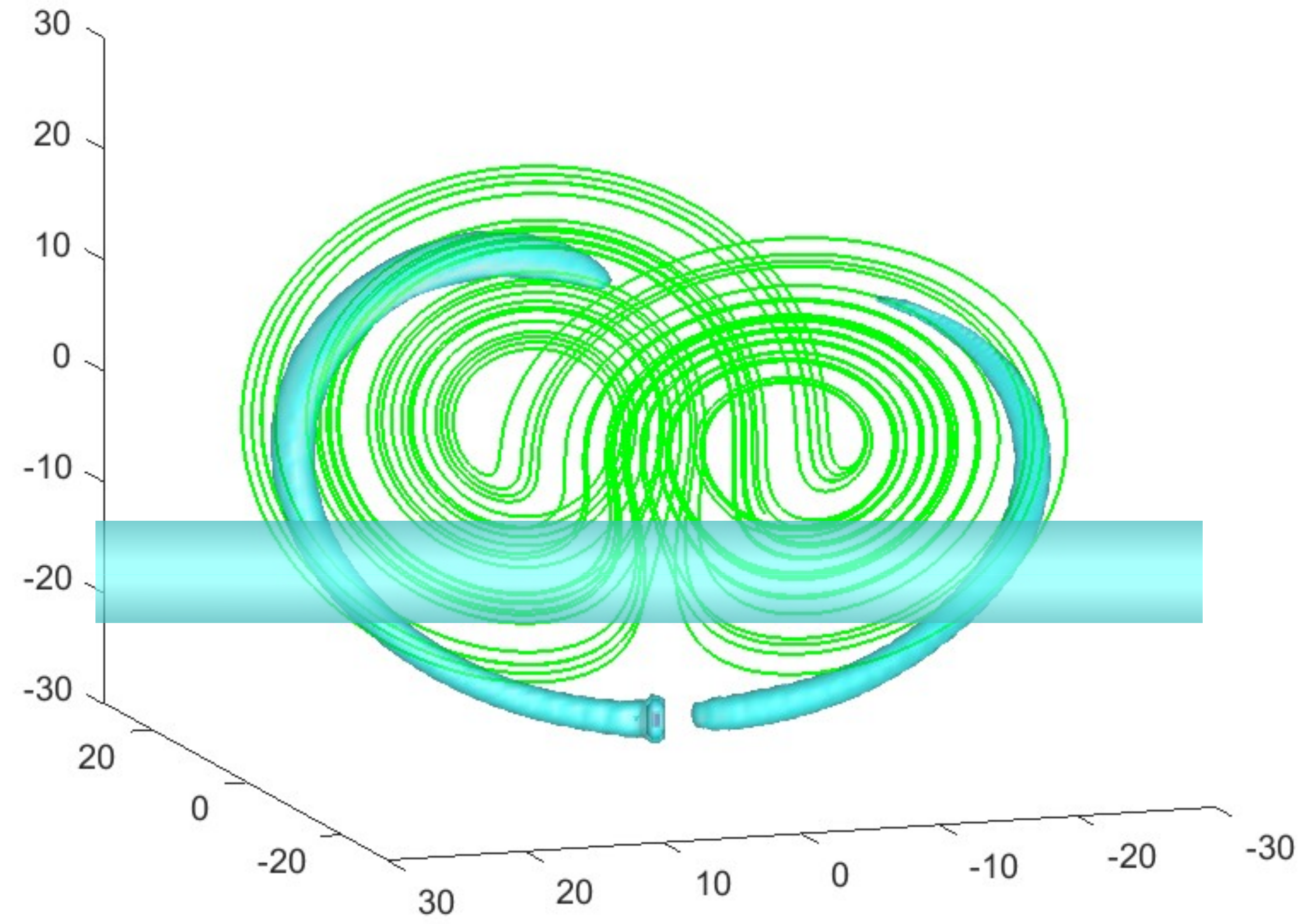

A priori × **Measurement**

Nonlinear Filter Comparison

Grid-based Bayesian Estimation Exploiting Sparsity (GBEES)

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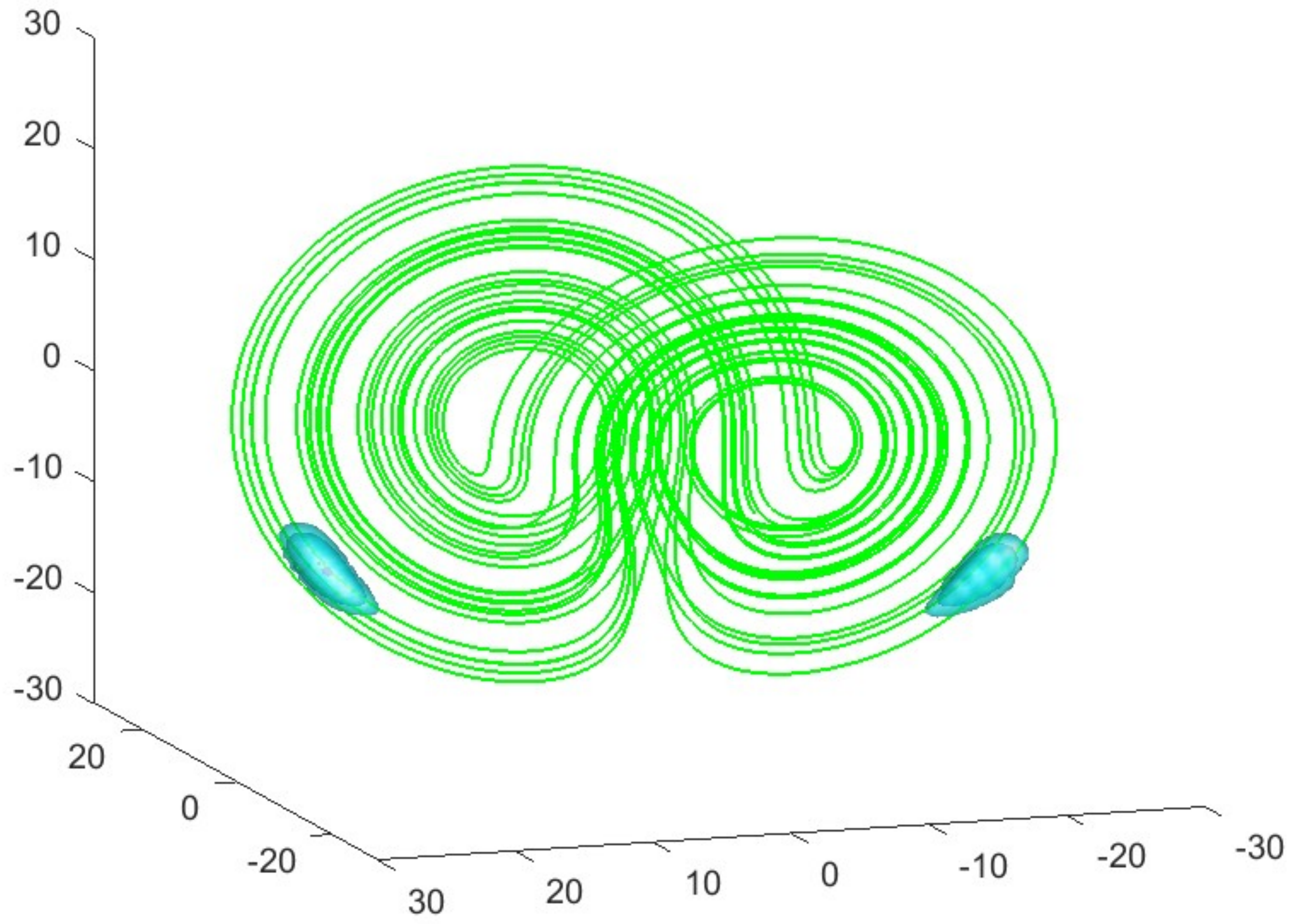


A priori × **Measurement** = **A posteriori**

Jovian Application: Three-Body Problem

Circular Restricted Three-Body Problem (CR3BP)

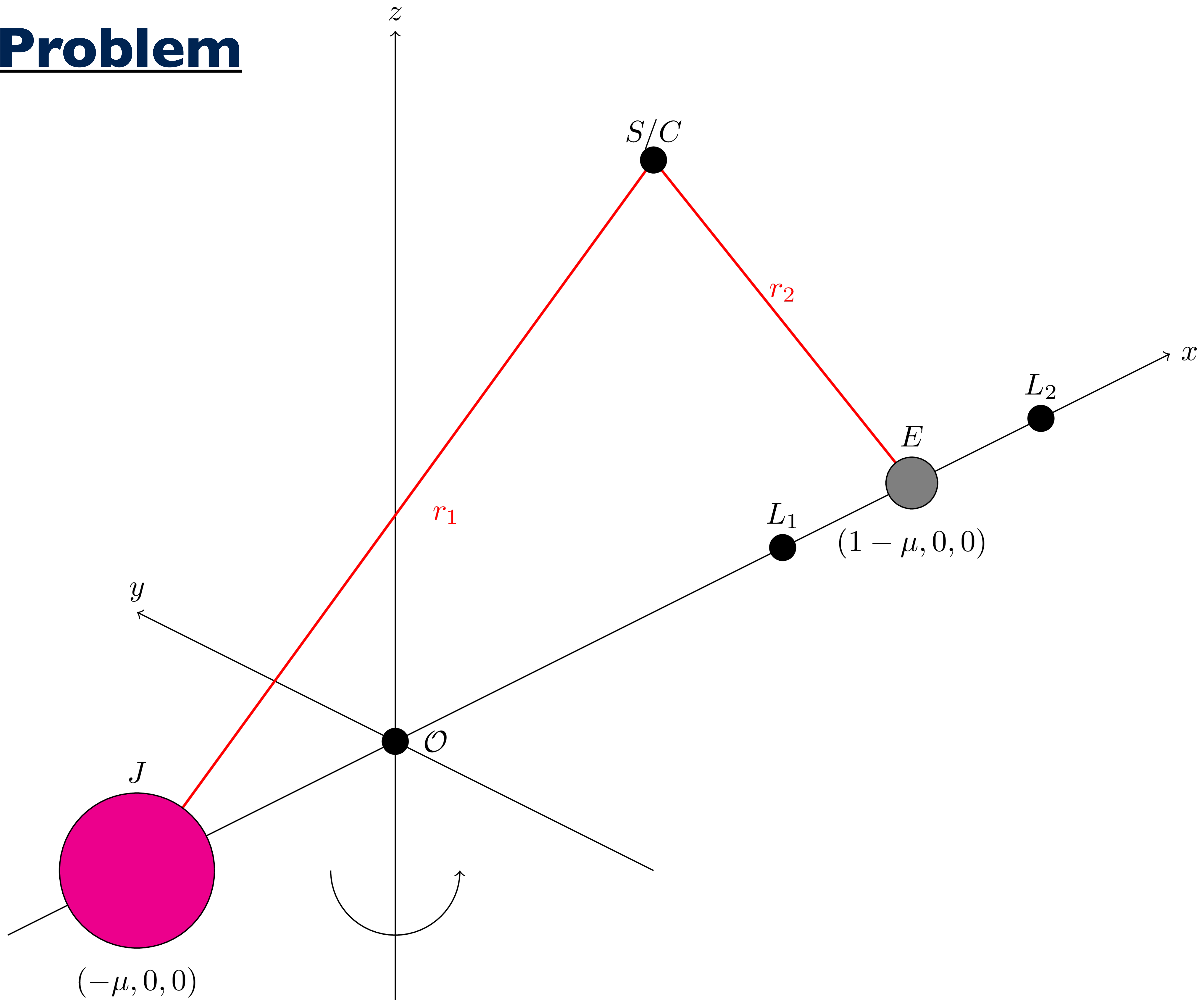
- We look to apply the developed framework to another systems applicable to Jovian trajectories

Circular Restricted Three-Body Problem

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix}, \quad \dot{\boldsymbol{x}} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ 2v_y + \Omega_x \\ -2v_x + \Omega_y \\ \Omega_z \end{bmatrix}$$

where $\Omega(x, y) = \frac{x^2 + y^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{\mu(1-\mu)}{2}$

✴ We use initial conditions generated from the JPL Three-Body Periodic Orbit Catalog for the Jupiter-Europa system

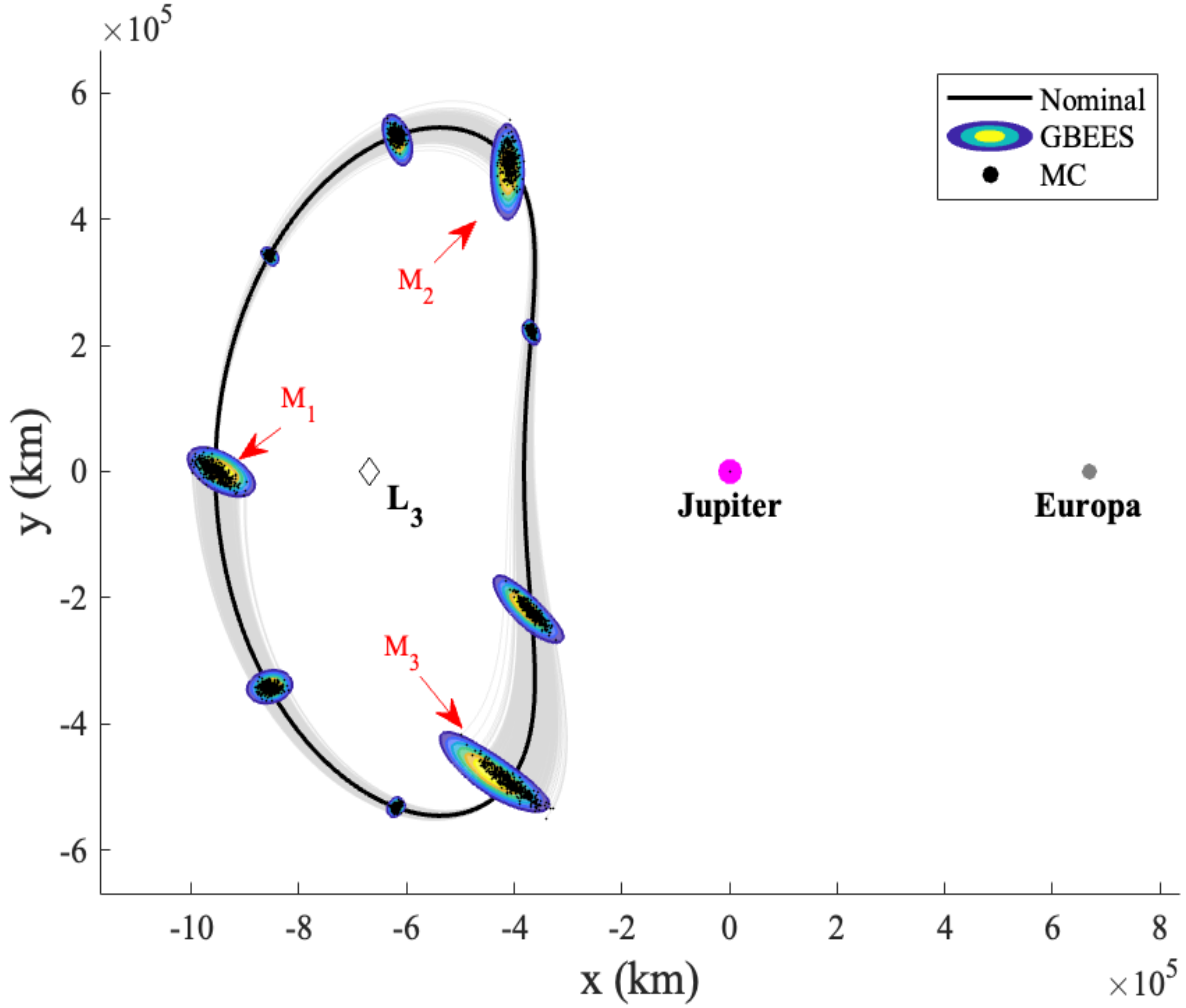


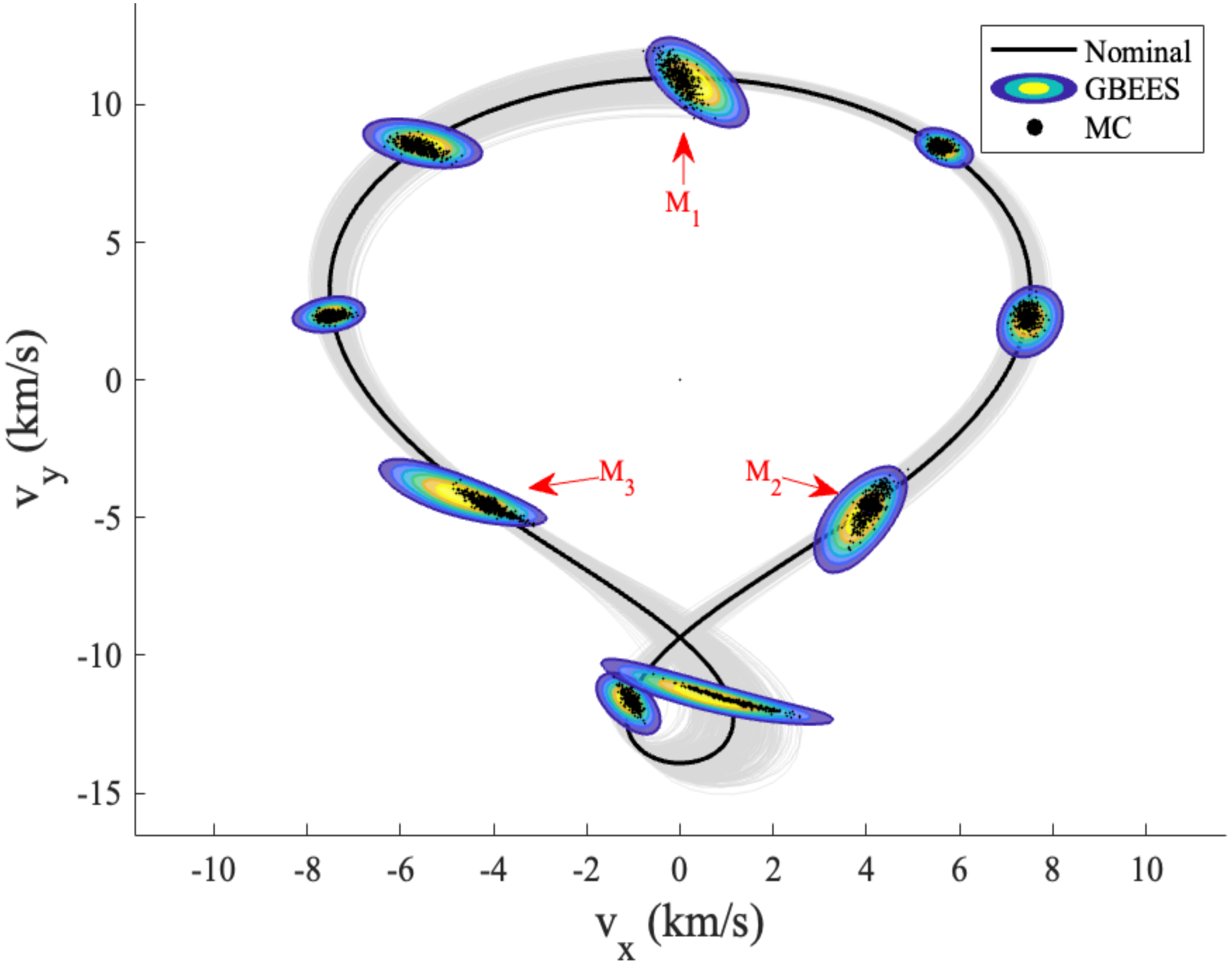
Park, Ryan. "Jet Propulsion Laboratory Three-Body Periodic Orbit Catalog." (2024)

Jovian Application: Lagrange Point Orbits

Review of measurement-sparse Jovian estimation

- Previous work applied a similar framework to **planar Lyapunov orbits** about L_3 in the Jupiter-Europa 3BP





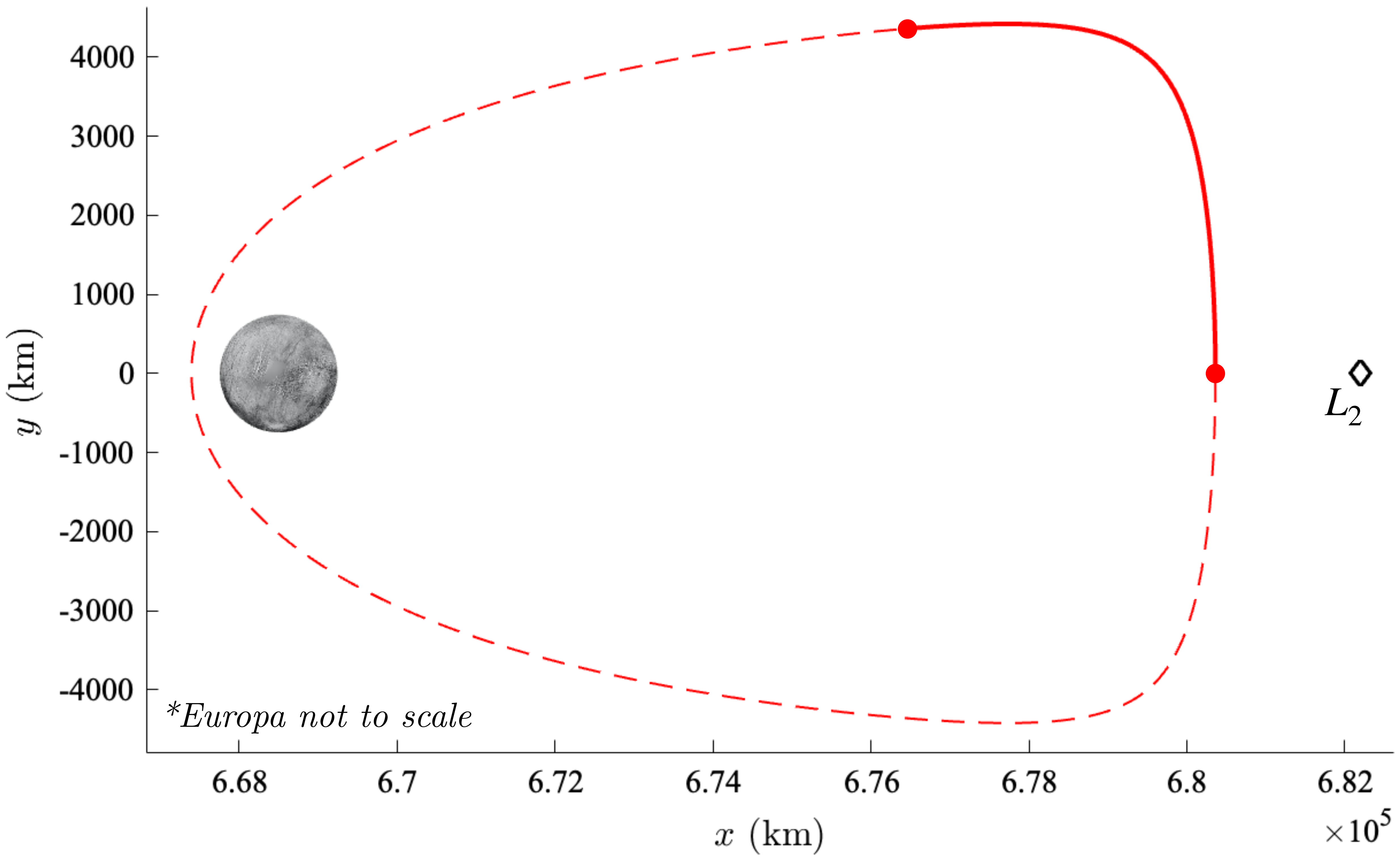
- We found that uncertainty remained near **Gaussian**, even with an infrequent measurement cadence (~1.17 days)

Hanson, Benjamin L., et al. "State Estimation of Chaotic Trajectories: A Higher-Dimensional, Grid-Based, Bayesian Approach to Uncertainty Propagation." AIAA. (2024)

Jovian Application: Low-Prograde Orbit

Revised framework applied to measurement-sparse Jovian estimation

- Implement linear filter estimation with new comparison framework on Jovian trajectory:
  - ✶ Initial condition resulting in eastern, low-prograde orbit about Europa
  - ✶ Propagated for 14 hours w/ RK8(7) and GBEES
  - ✶ No measurements and negligible process noise
  - ✶ α-convex hull comparison metric



$$\boldsymbol{x}_0 = \begin{bmatrix} x & \text{(km)} \\ y & \text{(km)} \\ v_x & \text{(m/s)} \\ v_y & \text{(m/s)} \end{bmatrix} = \begin{bmatrix} 6.803 \times 10^5 \\ 0 \\ 0 \\ 0.8623 \end{bmatrix}$$

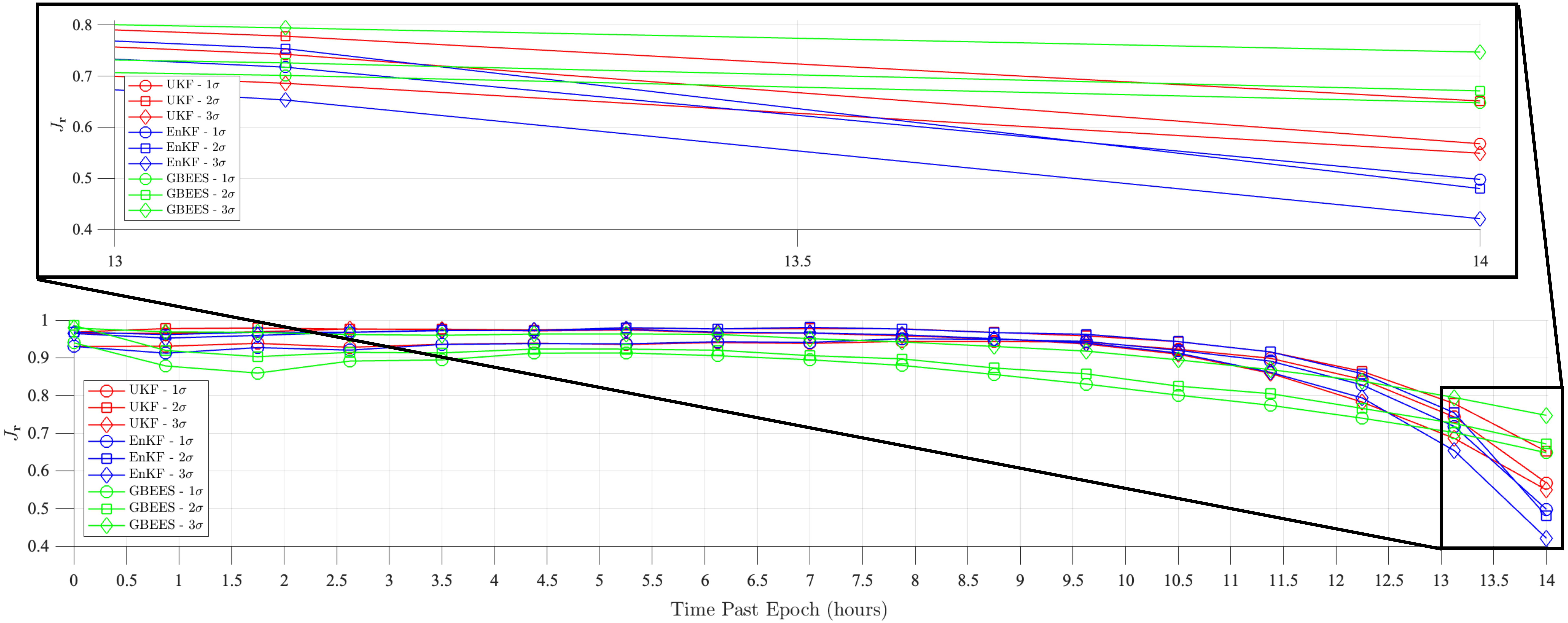
$$\sigma_r = 100 \text{ km}, \quad \sigma_v = 10 \text{ m/s}$$

Filter	Parameters
Particle Filter (truth)	Particles: 10^6
UKF	$\alpha = 10^{-3}, \beta = 2, \kappa = 0$
EnKF	Members: 10^4
GBEES	$p_{thresh} = 10^{-7}$

Park, Ryan. "Jet Propulsion Laboratory Three-Body Periodic Orbit Catalog." (2024)
Stastny, Nathan, et al. "Autonomous optical navigation at Jupiter: a linear covariance analysis." JSpRo. (2024)

Jovian Application: Low-Prograde Orbit

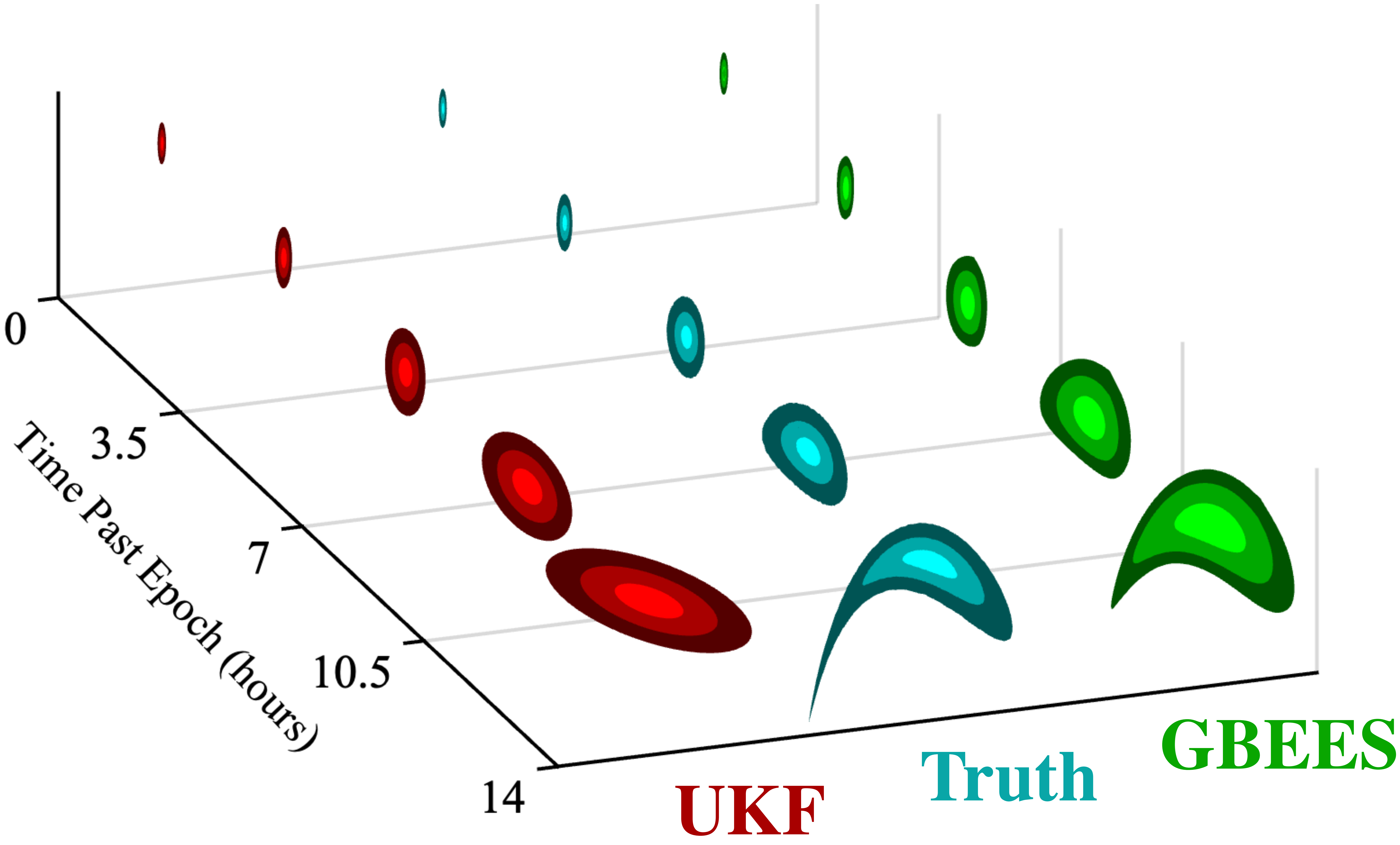
Comparing linear estimation with GBEES



J_r	UKF	EnKF	GBEES
1σ	0.5678	0.4978	**0.6479**
2σ	0.6514	0.4800	**0.6713**
3σ	0.5492	0.4209	**0.7472**

Jovian Application: Low-Prograde Orbit

Comparing linear estimation with GBEES

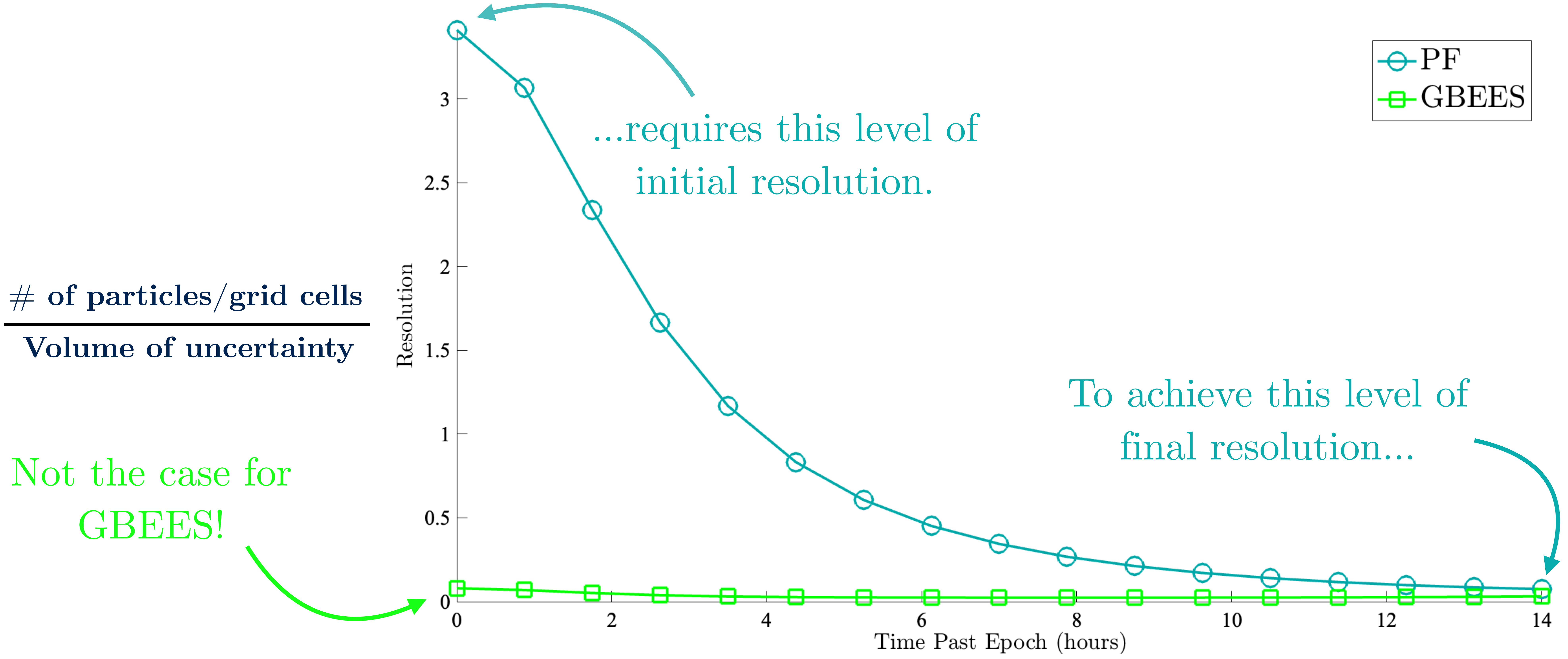


Jovian Application: Low-Prograde Orbit

Comparing Particle Filter with GBEES

- We utilize a high-resolution PF as a truth distribution, so why don't we use it for estimation?
  - ✴ To achieve **sufficient resolution** at a distant measurement epoch requires a large (**usually unknown**) number of particles that are marched from the previous epoch

$$\textbf{Resolution} = \frac{\#\ \textbf{of particles/grid cells}}{\textbf{Volume of uncertainty}}$$



  - ✴ GBEES nearly maintains resolution by **growing with the uncertainty**

Conclusion

Comments on Results and Future Work

- **Low-Europa Orbit**

J_r	UKF (Cartesian)	UKF (Equinoctial)	EnKF (Cartesian)	EnKF (Equinoctial)
1σ	N/A	**0.1763**	0.0445	0.1727
2σ	N/A	**0.1414**	0.0427	0.1237
3σ	N/A	**0.1049**	0.0366	0.0800

  * 1σ position uncertainty estimated by the UKF (Equinoctial) is able to maintain $J_r \geq 0.5$ compared with truth distribution for nearly 2 revolutions without measurements, with local minima located at periapsis

- **Low-Prograde Orbit in Jupiter-Europa Three-Body System**

J_r	UKF	EnKF	GBEES
1σ	0.5678	0.4978	**0.6479**
2σ	0.6514	0.4800	**0.6713**
3σ	0.5492	0.4209	**0.7472**

  * While linear filters are able to estimate uncertainty better when distributions are near-Gaussian, GBEES is more accurate when distributions are far from Gaussian, which occurs in about 14 hours for the given LPO

- **Future Work**
  * Propagating in the **slow-changing**, three-body local orbit elements
  * **Parallelization** of Riemann solver embedded within GBEES
  * Dynamics sourced from an **ephemeris-level** numerical integrator

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All code can be found at: https://github.com/bhanson10/GBEES and https://github.com/bhanson10/KePASSA2024

Thank you for your time. Questions?