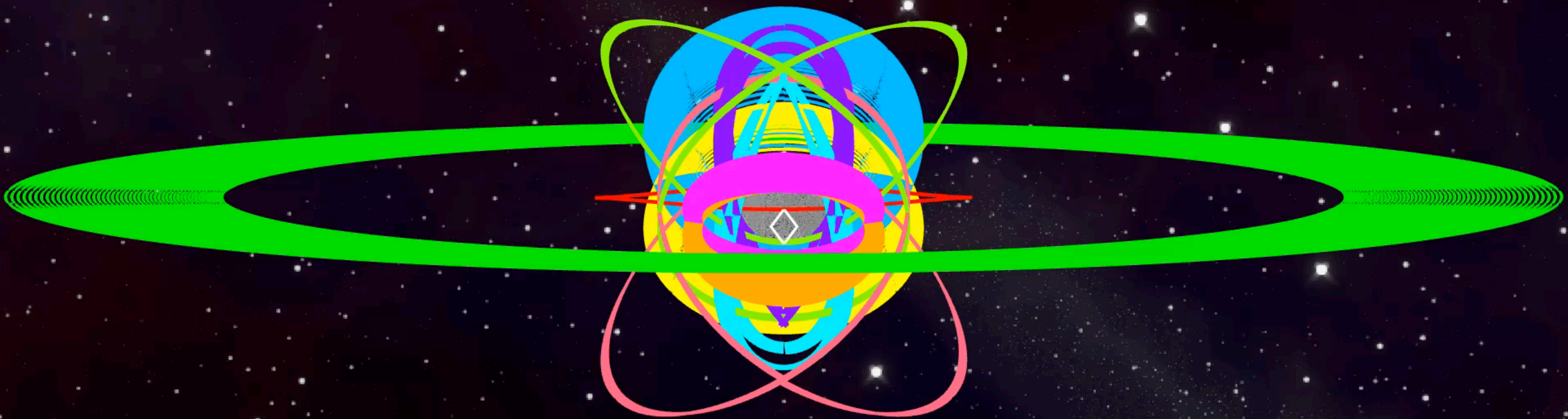
Efficient Prediction of the Gaussianity Validity Time in the Circular Restricted Three-Body Problem



COPERNAICUS Seminar - Sept. 22, 2025

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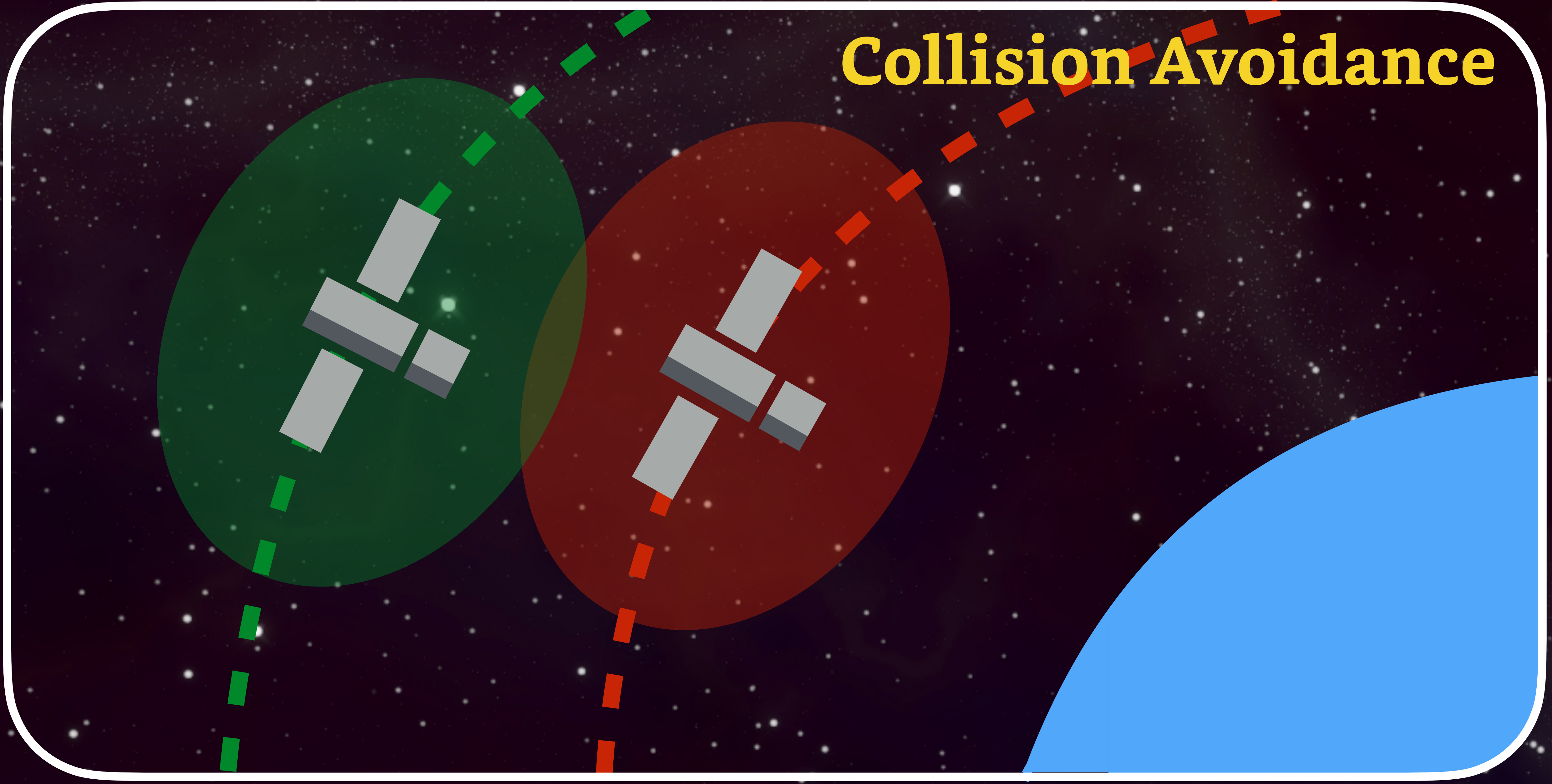
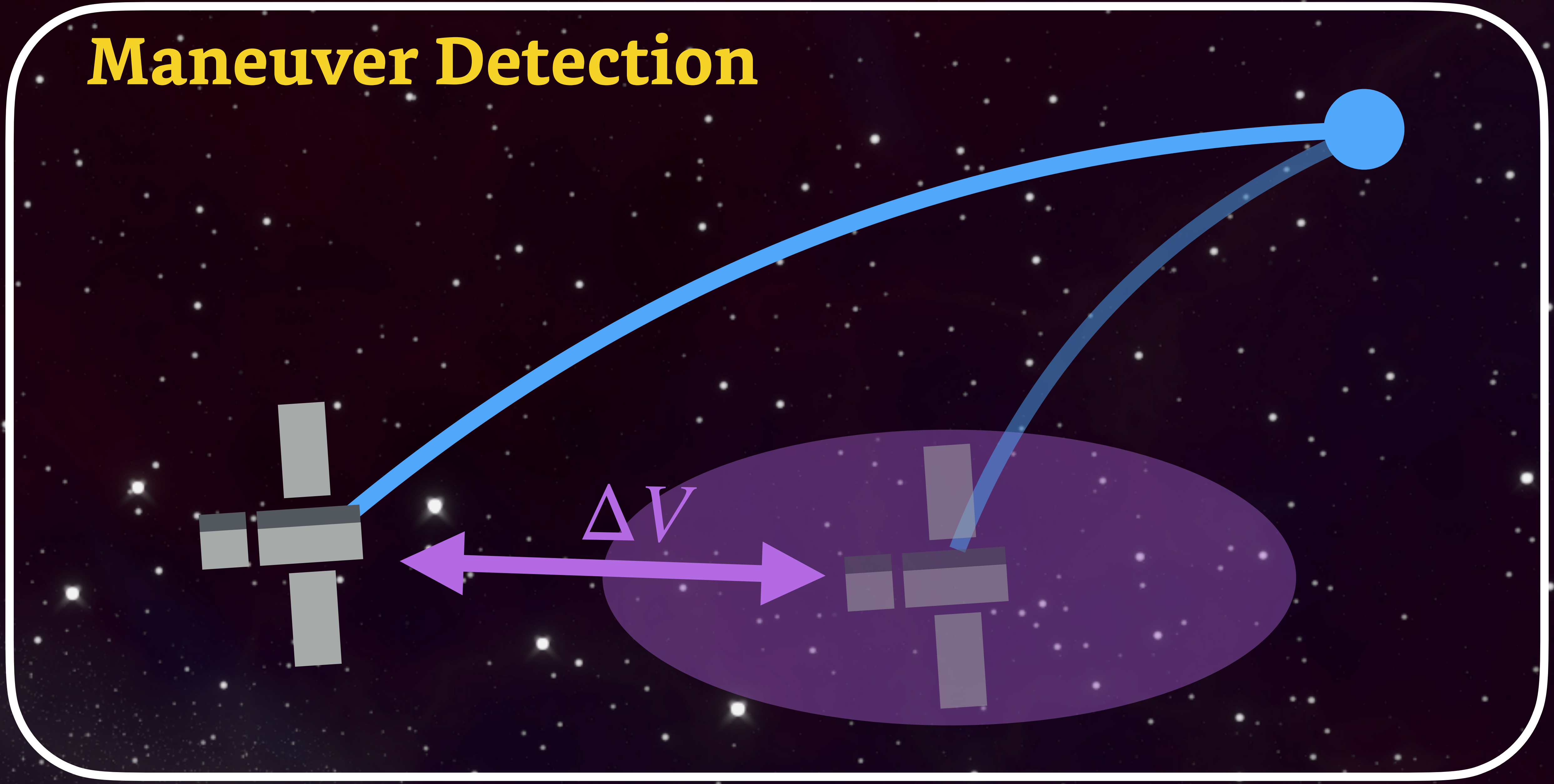
The Gaussian assumption is ubiquitous in SDA...





...but where is the Gaussian validation?

The Nonlinear State Estimation Problem

- Consider the state estimation of a general system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t) + \boldsymbol{w}, \quad \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}, t) + \boldsymbol{v}$$

- If $\boldsymbol{f}, \boldsymbol{h}$ are linear and $\boldsymbol{w}, \boldsymbol{v}$ are Gaussian zero-mean white noise, then

$$\boldsymbol{X}(t) \sim \mathcal{N}\left(\boldsymbol{x} \,|\, \boldsymbol{\mu}(t), \boldsymbol{\Sigma}(t)\right) = \frac{1}{\sqrt{(2\pi)^d |\boldsymbol{\Sigma}(t)|}} \exp\left(-\frac{1}{2}\left(\boldsymbol{x} - \boldsymbol{\mu}(t)\right)^\top \boldsymbol{\Sigma}(t)^{-1}\left(\boldsymbol{x} - \boldsymbol{\mu}(t)\right)\right)$$

- However, if $\boldsymbol{f}, \boldsymbol{h}$ are nonlinear, then generally speaking

$$\boldsymbol{X}(t) \sim p(\boldsymbol{x}, t) \neq \mathcal{N}\left(\boldsymbol{x} \,|\, \boldsymbol{\mu}(t), \boldsymbol{\Sigma}(t)\right)$$

Fundamental Questions

1. How do we measure Gaussianity?
2. How long does it take for state uncertainty to become non-Gaussian?
3. Can we predict when state uncertainty is becoming non-Gaussian with an abstraction more efficient to propagate than a dense Monte Carlo?

Being "kind-of" Gaussian

Analytical vs. Statistical Definitions

> "Being 'kind-of' Gaussian is like being 'kind-of' dead."
>
> *-Dr. Tom Bewley, UCSD*

Analytical Definition of a Gaussian

$$p(\boldsymbol{x} \mid \boldsymbol{\mu}; \boldsymbol{\Sigma}) = \frac{1}{\sqrt{(2\pi)^d \, |\boldsymbol{\Sigma}|}} \exp\left(-\frac{1}{2} (\boldsymbol{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu}) \right)$$

Statistical Definition of a Gaussian

A Monte Carlo comparison of the Type I and Type II error rates of tests of multivariate normality

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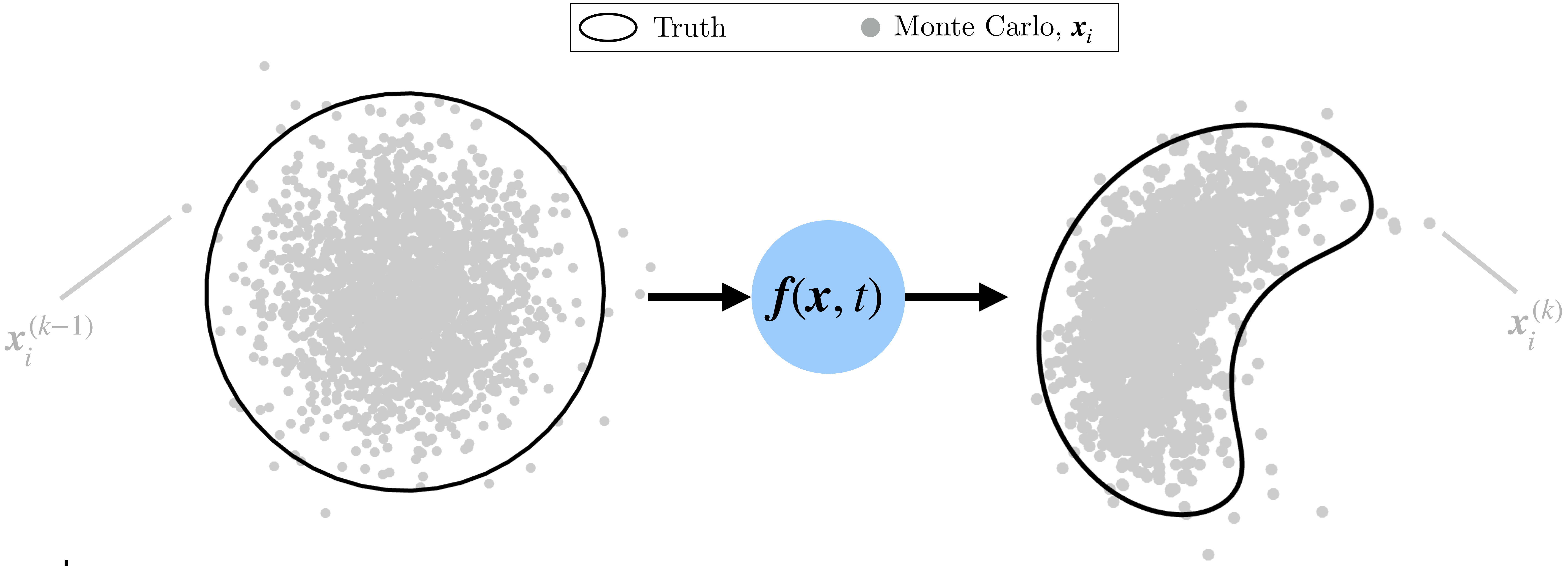
Table 1. Tests of MVN.

Test	Class	Iris setosa
Mardia's skewness	Skewness/kurtosis	Do not reject
Mardia's kurtosis	Skewness/kurtosis	Do not reject
Hawkins	Goodness-of-fit	Reject
Koziol	Goodness-of-fit	Do not reject
Mardia–Foster	Skewness/kurtosis	Reject
Royston	Goodness-of-fit	Reject
PRS	Goodness-of-fit	Do not reject
Henze–Zirkler	Consistent	Do not reject
Mardia–Kent	Skewness/kurtosis	Do not reject
Romeu–Ozturk	Goodness-of-fit	Reject
Singh (classical)	Graphical/Correlational	Reject
Singh (robust)	Graphical/Correlational	Reject
MSL	Goodness-of-fit	Do not reject

Henze-Zirkler Statistic

$d = 2$
$n = 2000$



$$\mathrm{HZ} = \left[\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\exp\left(-\frac{\beta^2}{2}D_{ij}\right)\right] - \left[2\left(1+\beta^2\right)^{-\frac{d}{2}}\sum_{i=1}^{n}\exp\left(-\frac{\beta^2}{2(1+\beta^2)}D_i\right)\right] + \left[n(1+2\beta^2)^{-\frac{d}{2}}\right]$$

- d = dimensionality
- n = # of Monte Carlo samples
- $\beta = \frac{1}{\sqrt{2}}\left(\frac{n(2d+1)}{4}\right)^{\frac{1}{d+4}}$, smoothing parameter
- $D_{ij} = \left(\boldsymbol{x}_i - \boldsymbol{x}_j\right)^\top \Sigma^{-1}\left(\boldsymbol{x}_i - \boldsymbol{x}_j\right)$, Mahalanobis distance between each point and every other point
- $D_i = \left(\boldsymbol{x}_i - \boldsymbol{\mu}\right)^\top \Sigma^{-1}\left(\boldsymbol{x}_i - \boldsymbol{\mu}\right)$, Mahalanobis distance between each point and the mean

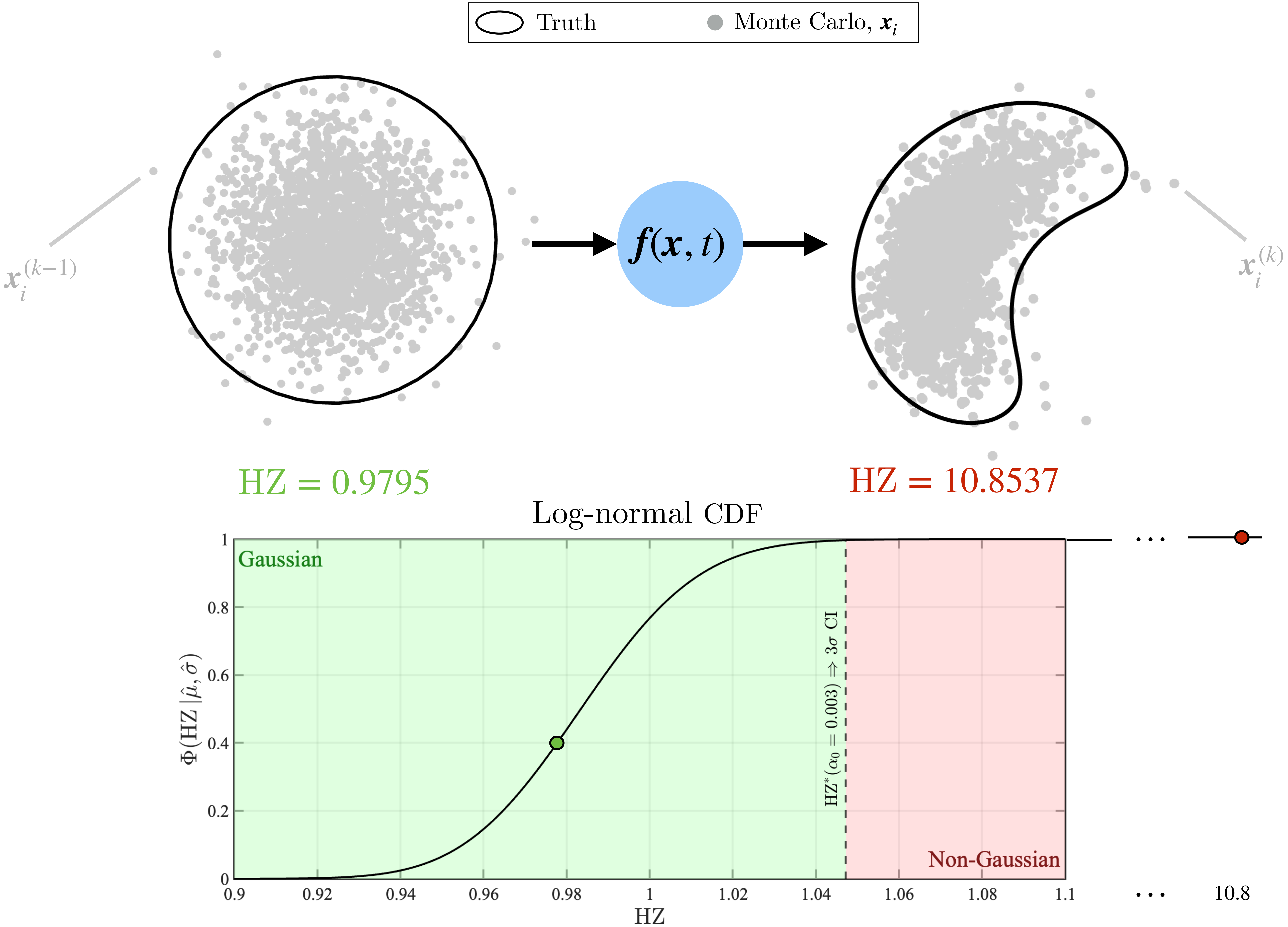
$\mathrm{HZ} > \mathrm{HZ}^*(\alpha_0 = 0.003) \quad \Rightarrow \quad H_0$ should be rejected
$\mathrm{HZ} \leq \mathrm{HZ}^*(\alpha_0 = 0.003) \quad \Rightarrow \quad H_0$ cannot be rejected

- HZ* = HZ Gaussian Validity Boundary (GVB)

- HZ is approximately log-normally distributed, so a null hypothesis H_0 of Gaussianity may be tested

Henze-Zirkler Statistic

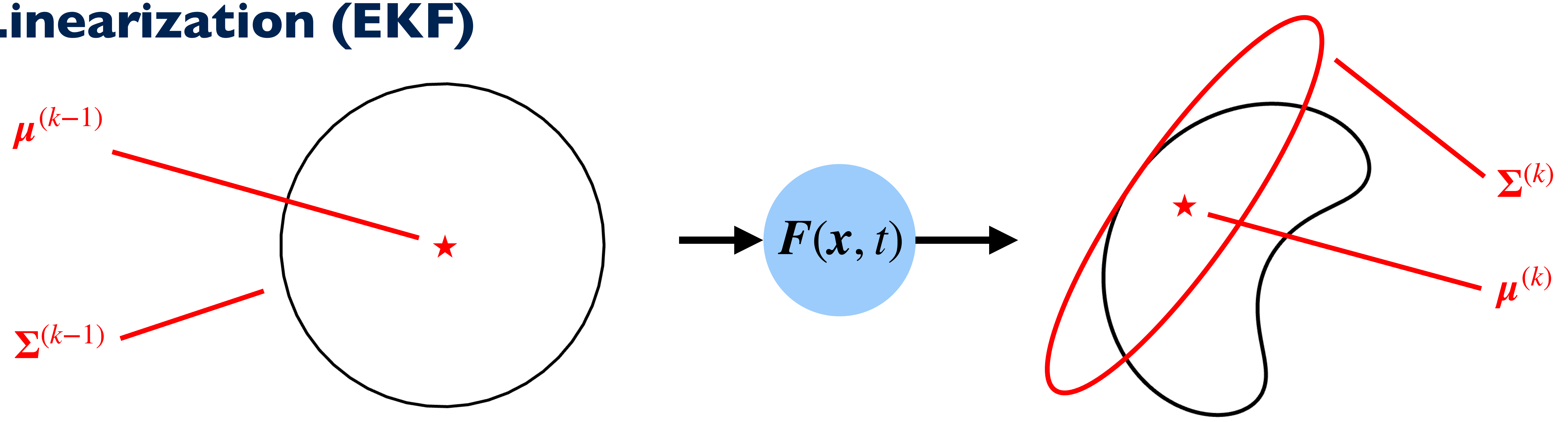
$d = 2$
$n = 2000$



The Unscented Transform

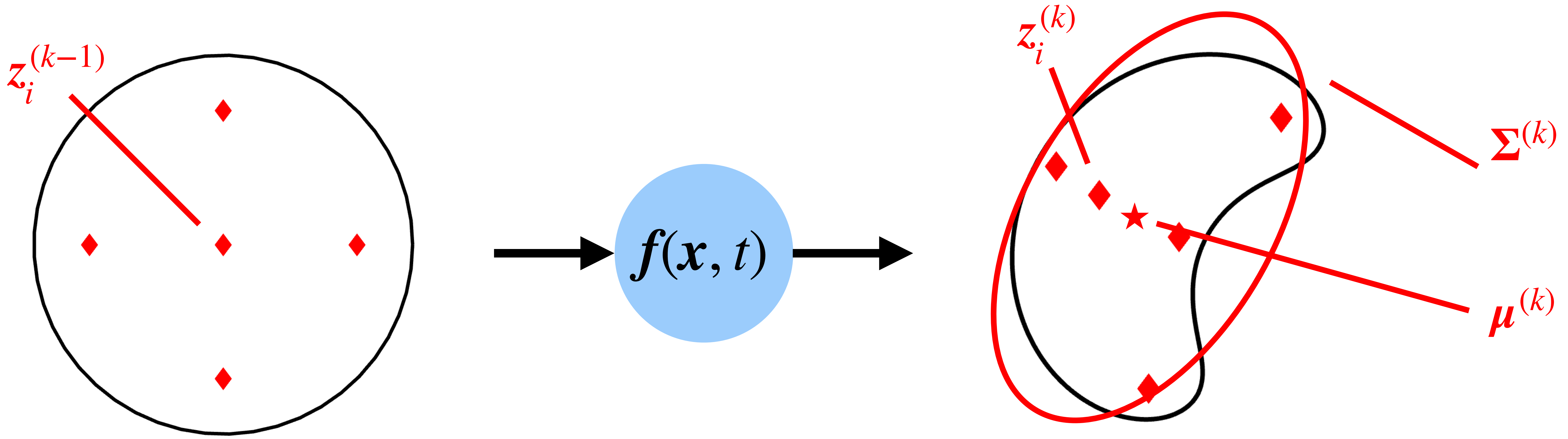
> "…it is easier to approximate a probability distribution than it is to approximate an arbitrary nonlinear function…"
>
> *-Dr. Jeffrey Uhlmann, Inventor of the Unscented Transform*

Analytical Linearization (EKF)

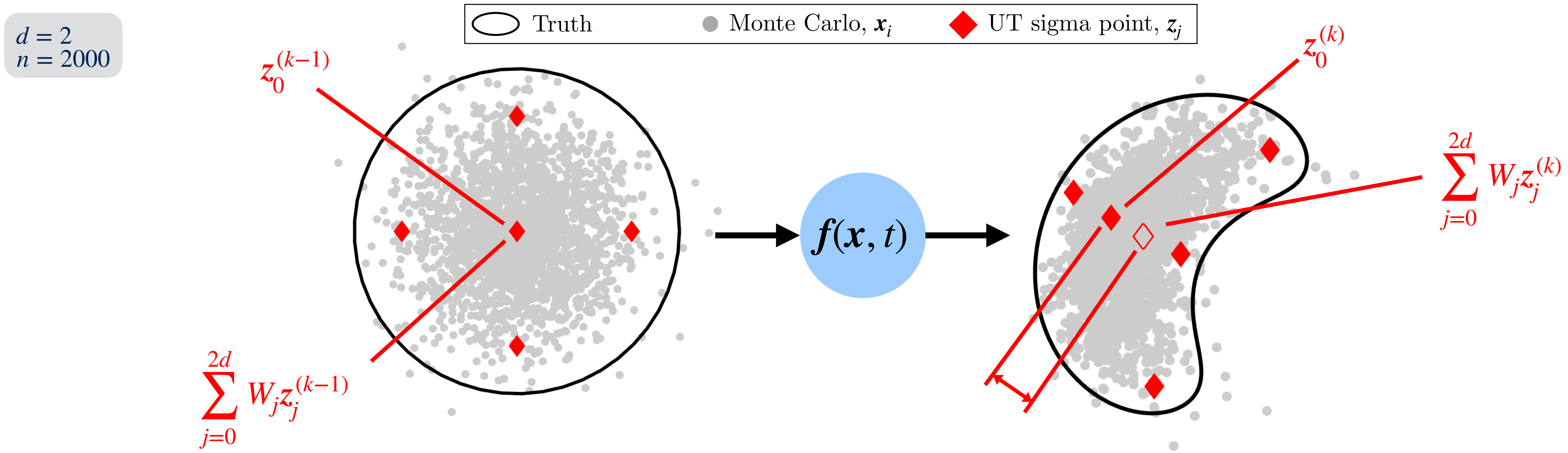


Statistical Linearization (UKF)

States	Weights
$z_0 = \boldsymbol{\mu}$	$W_0 = \kappa/(d+\kappa)$
$z_i = \boldsymbol{\mu} + \left(\sqrt{(d+\kappa)\Sigma}\right)_i$	$W_i = \kappa/\left(2(d+\kappa)\right)$
$z_{i+d} = \boldsymbol{\mu} - \left(\sqrt{(d+\kappa)\Sigma}\right)_i$	$W_{i+d} = \kappa/\left(2(d+\kappa)\right)$



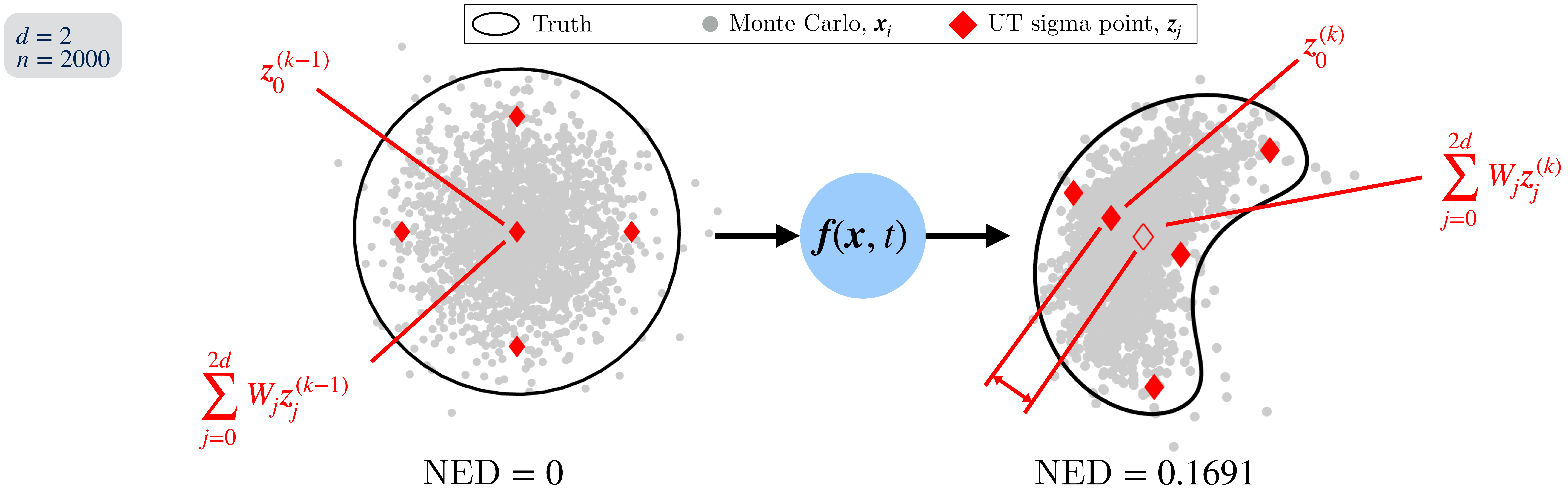
Normalized Euclidean Distance



$$\text{NED} = \left\| L^{-1} \left(z_0 - \sum_{j=0}^{2d} W_j z_j \right) \right\|$$, where L^{-1} is the inverse lower triangular of the covariance

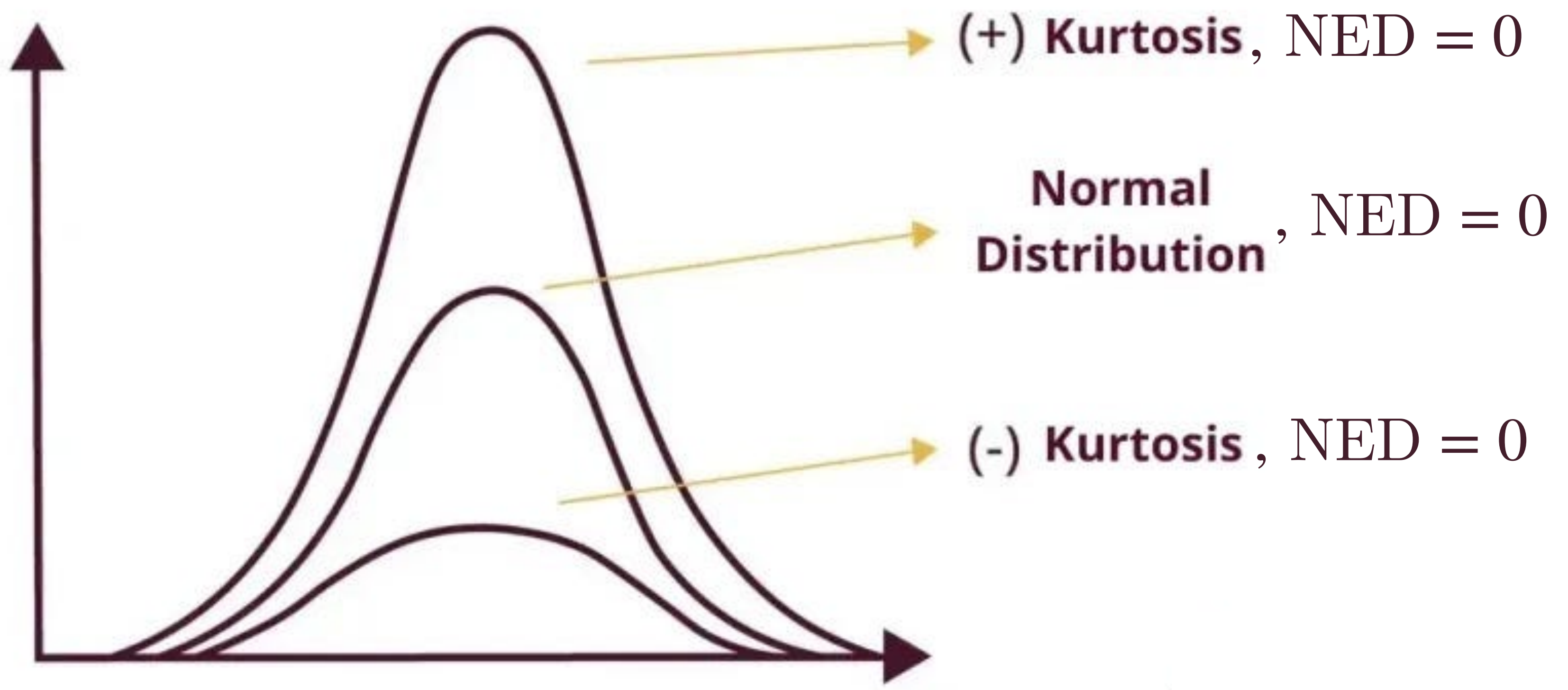
- NED may be calculated from the UT sigma points alone, meaning it requires a fraction of the samples that the HZ requires for an accurate value
- When $\boldsymbol{f}(\boldsymbol{x}, t)$ is linear, NED remains at 0; when $\boldsymbol{f}(\boldsymbol{x}, t)$ is nonlinear, NED may drift

Normalized Euclidean Distance



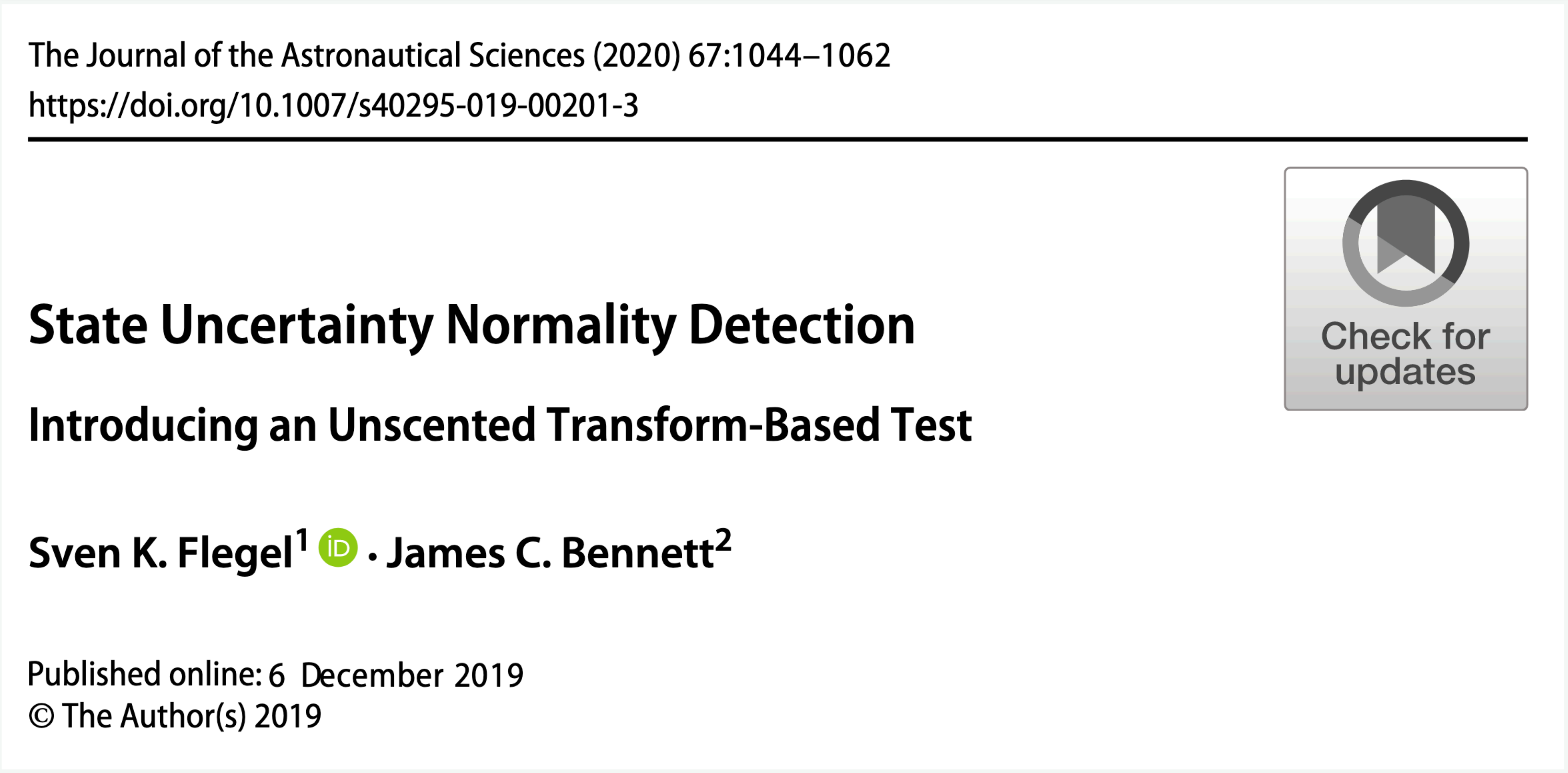
- NED is not a consistent statistical test; without mapping it to a consistent statistical test (HZ) we have no absolute information on the likelihood that the sigma points come from a Gaussian distribution

- **What about kurtosis?**



Two-Body Problem HZ-NED Mapping

- State uncertainty in closed orbits tends to oscillate between near-Gaussian during the quiescent, rectilinear phases and highly non-Gaussian near periapsis, as demonstrated in Flegel and Bennett*


The Journal of the Astronautical Sciences (2020) 67:1044–1062
https://doi.org/10.1007/s40295-019-00201-3

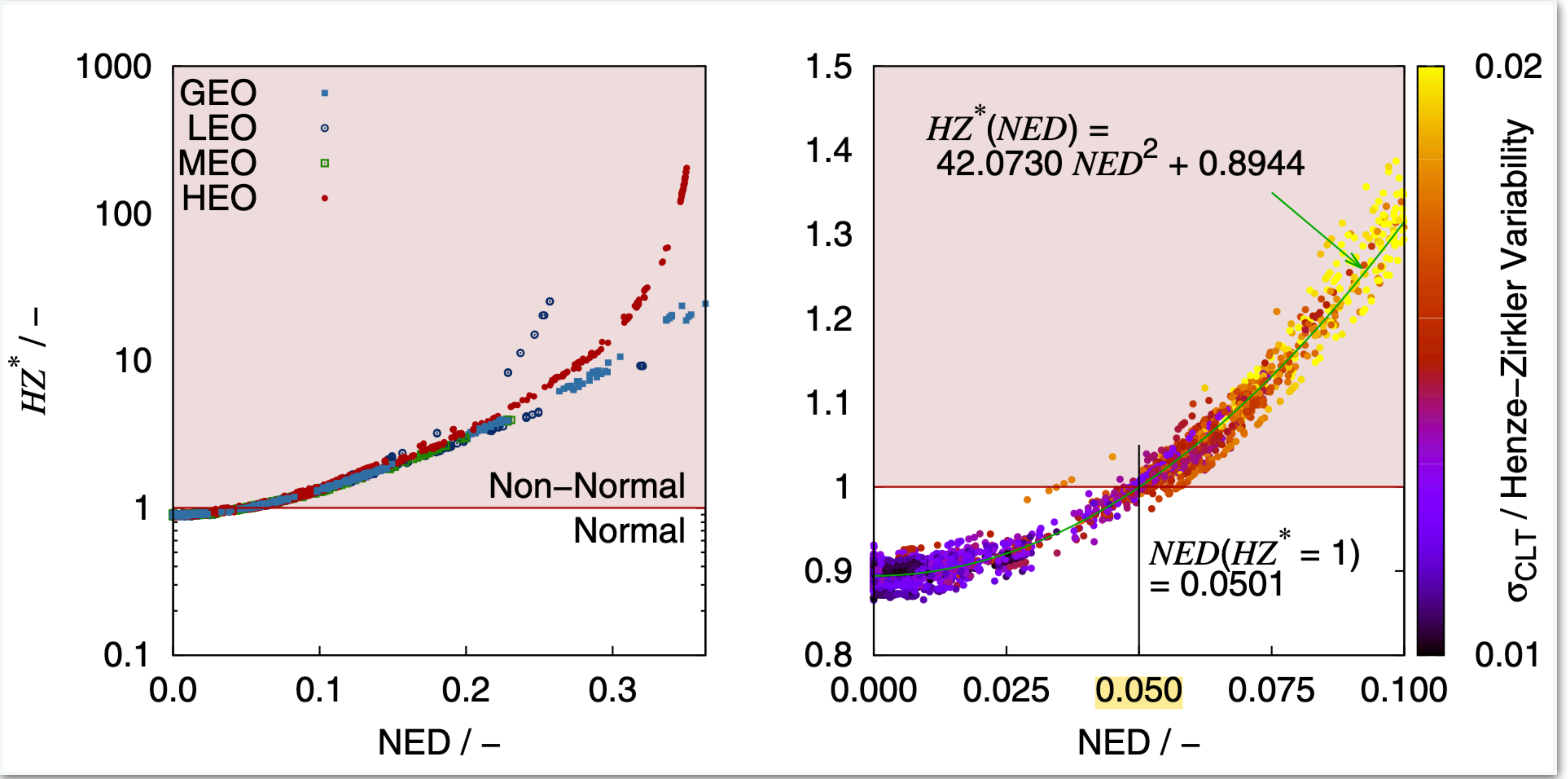
Check for updates

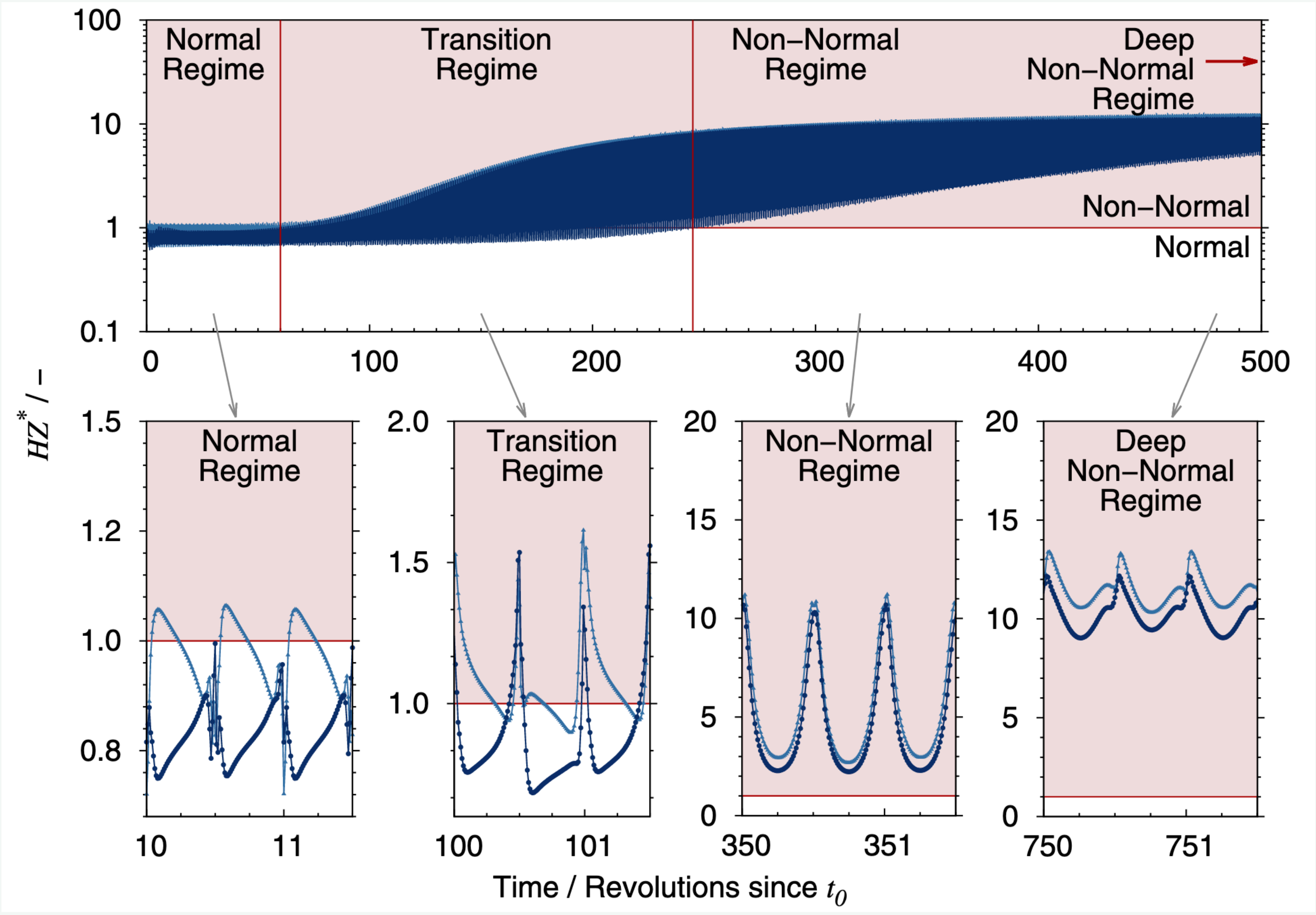
State Uncertainty Normality Detection

Introducing an Unscented Transform-Based Test

Sven K. Flegel[1] · James C. Bennett[2]

Published online: 6 December 2019



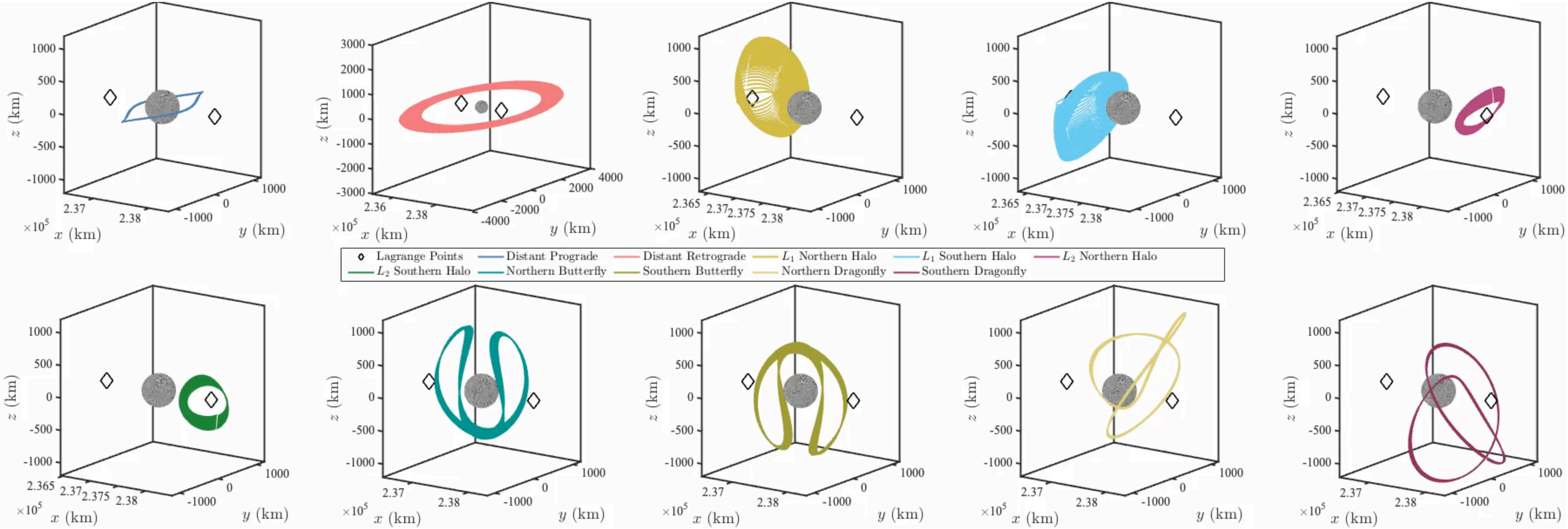


*Flegel, S.K. and Bennett, J.C., 2020. State Uncertainty Normality Detection: Introducing an Unscented Transform-Based Test. The Journal of the Astronautical Sciences 67(3), pp.1044-1062.

Saturn-Enceladus CR3BP Periodic Orbits

Objective: Determine the relationship (?) between HZ and NED in the Circular Restricted Three-Body Problem using periodic orbit families from the Saturn-Enceladus system.

Using 50 initial conditions from each of the ten following periodic orbit families...

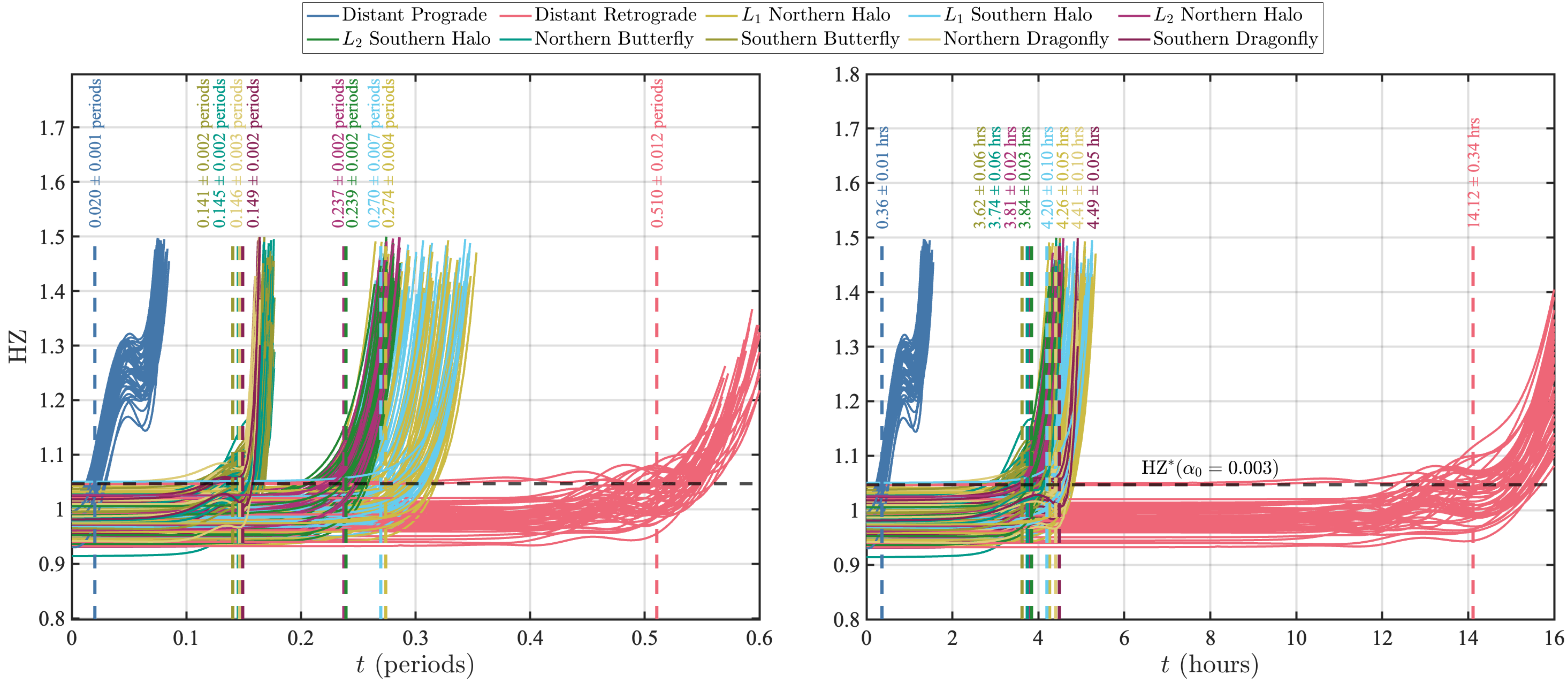


...map the HZ to the NED for the CR3BP.

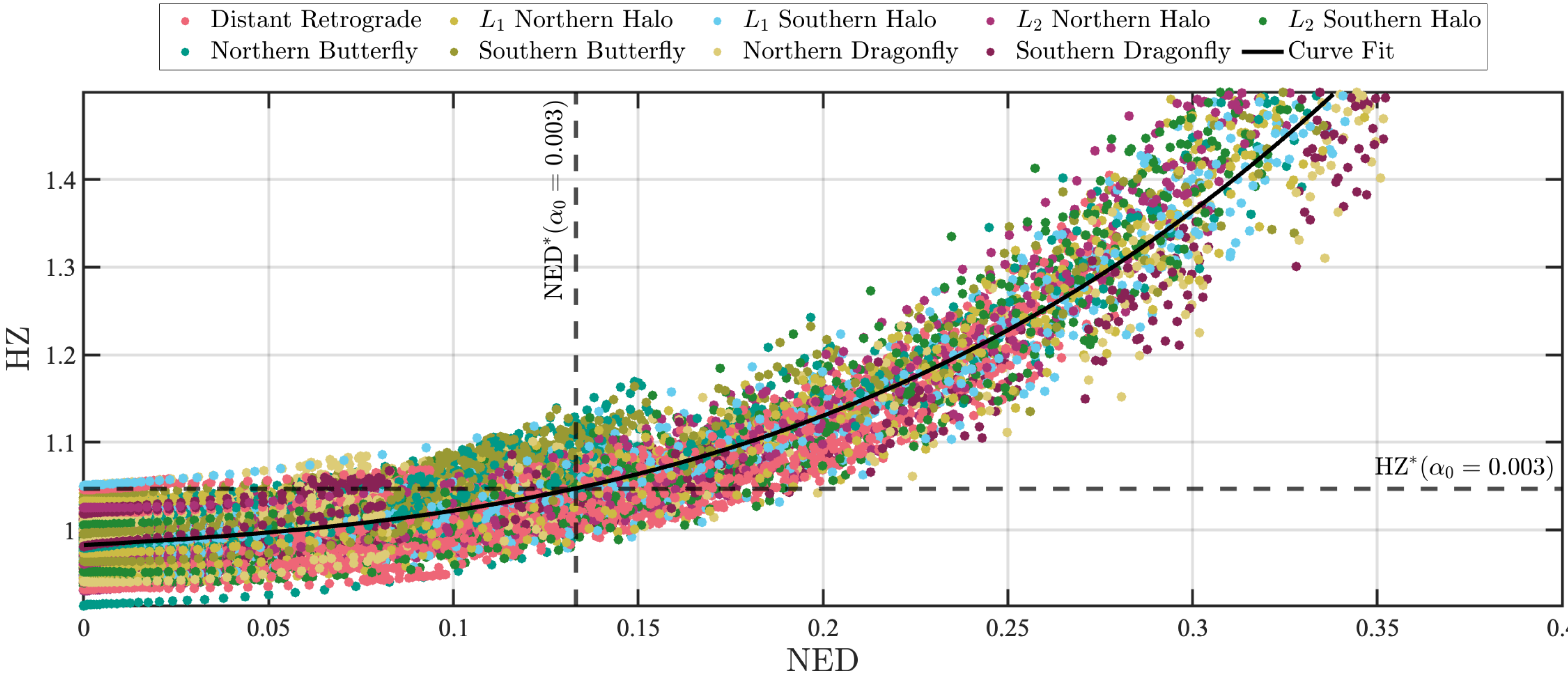
Time to Non-Gaussianity in CR3BP

Saturn-Enceladus Periodic Orbit Families

- Parameters:
  - 50 initial conditions per family, 5,000 random samples per initial condition,
  - Initial uncertainty: $\sigma_r = 1$ km, $\sigma_v = 1$ cm/s
  - $\mathrm{HZ}^* = q(\alpha_0 = 0.003) \Rightarrow 3\sigma$ confidence interval



Mapping HZ to NED for Saturn-Enceladus CR3BP



- Curve fit function (3σ confidence intervals):

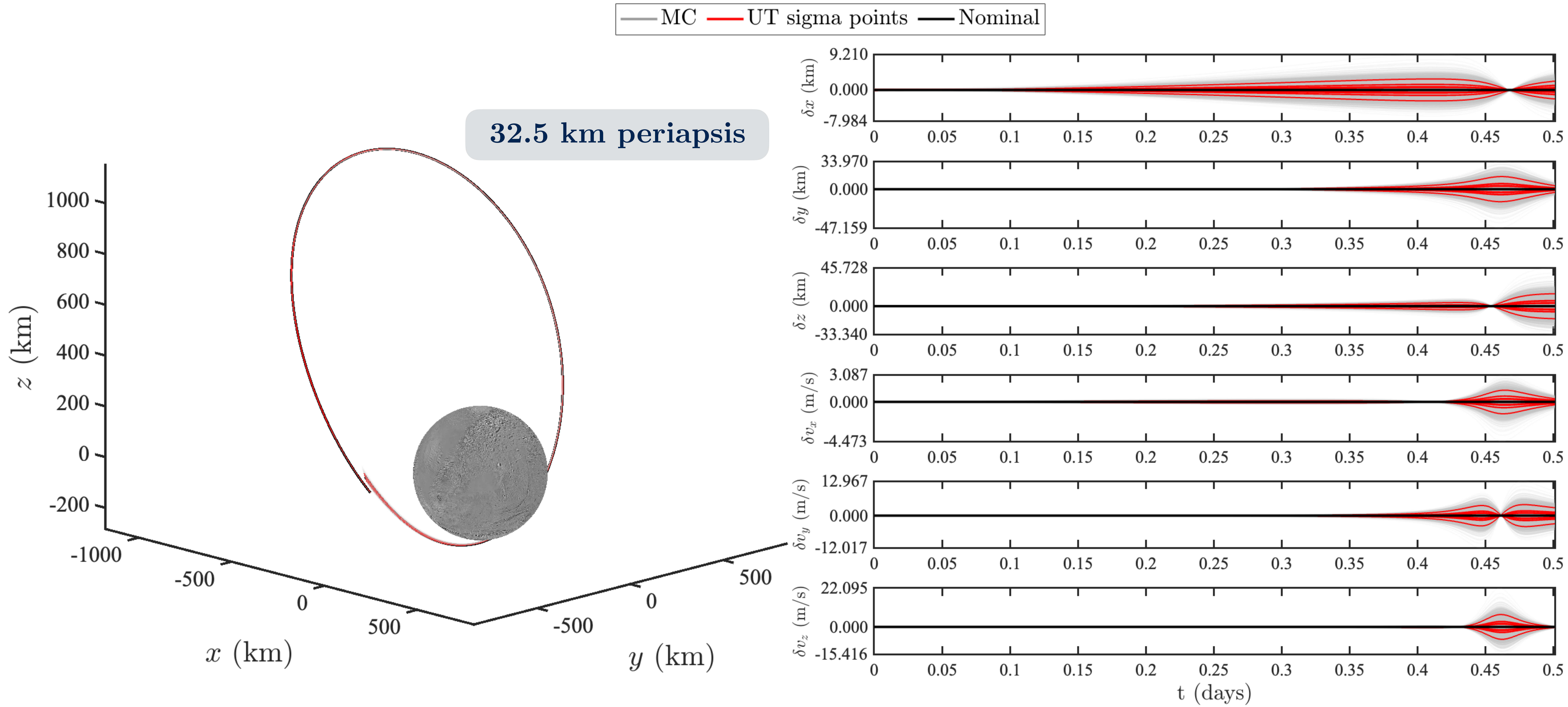
$$\mathrm{HZ}(\mathrm{NED}) = (9.308 \pm 0.864)\mathrm{NED}^3 + (0.669 \pm 0.354)\mathrm{NED}^2 + (0.232 \pm 0.035)\mathrm{NED} + (0.983 \pm 0.001)$$

- NED GVB:

$$\mathrm{NED}^*(\alpha_0 = 0.003) = 0.1330 \pm 0.002$$

Utilization of NED GVB For UT-only GVT Prediction

- First UT-only GVT prediction: **Highly Inclined Saturn-Enceladus Halo Orbit** from Russell and Lara*



Highly Inclined Saturn-Enceladus halo orbit propagated for 0.5 days, with initial uncertainty $\sigma_r = 100$ m and $\sigma_v = 1$ cm/s.

*Russell, R.P. and Lara, M., 2009. On the design of an Enceladus science orbit. Acta Astronautica, 65(1-2), pp.27-39.

Utilization of NED GVB For UT-only GVT Prediction

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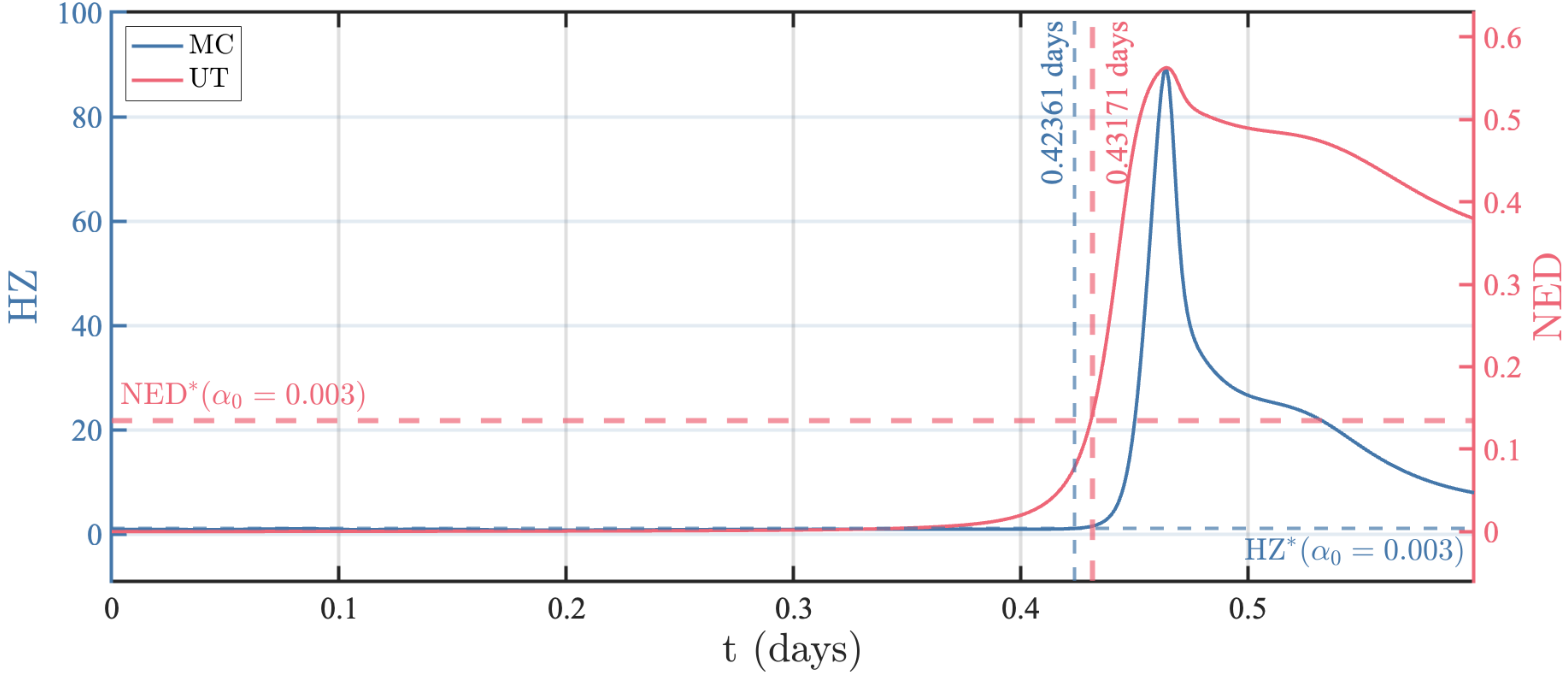
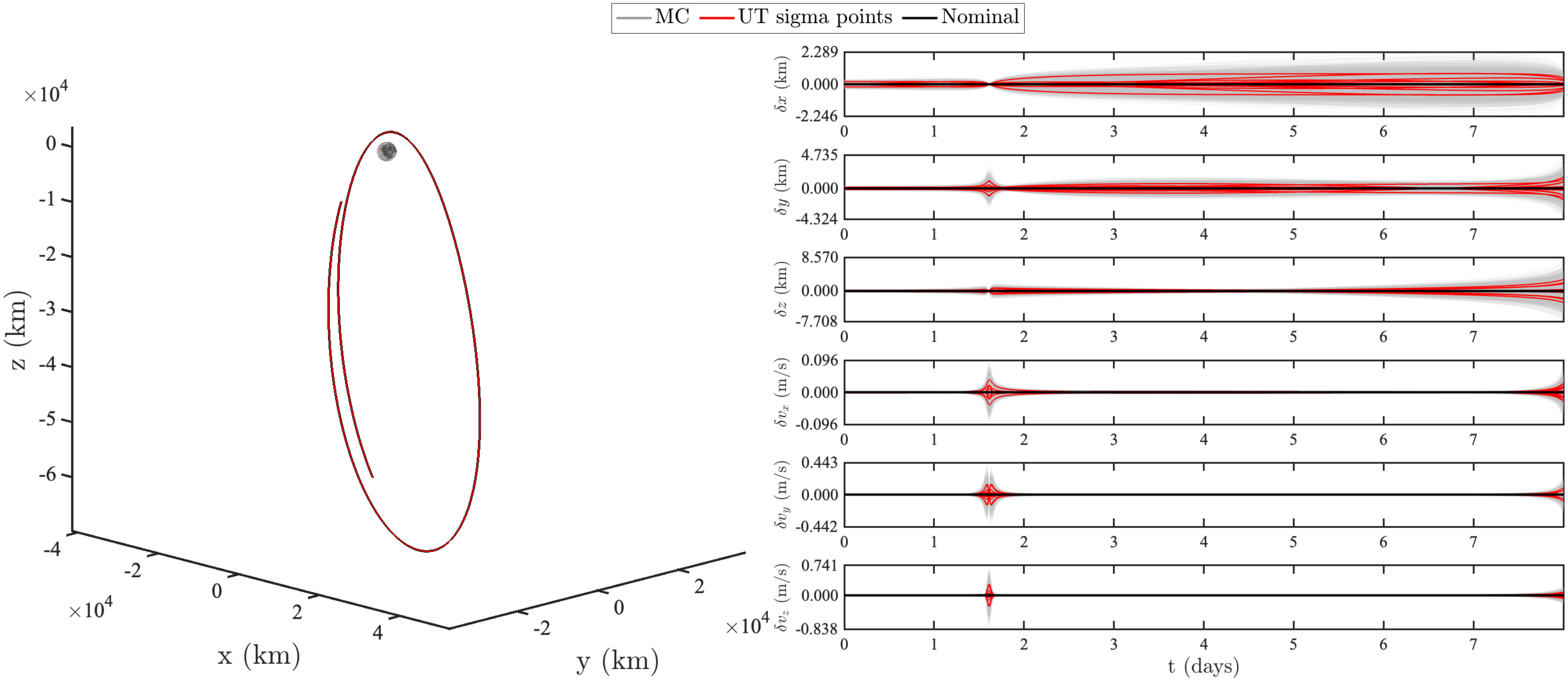


UT-only GVT prediction is within 11.664 minutes of a MC-based GVT prediction on a completely new trajectory using our derived NED*

**Russell, R.P. and Lara, M., 2009. On the design of an Enceladus science orbit. Acta Astronautica, 65(1-2), pp.27-39.*

Utilization of NED GVB For UT-only GVT Prediction

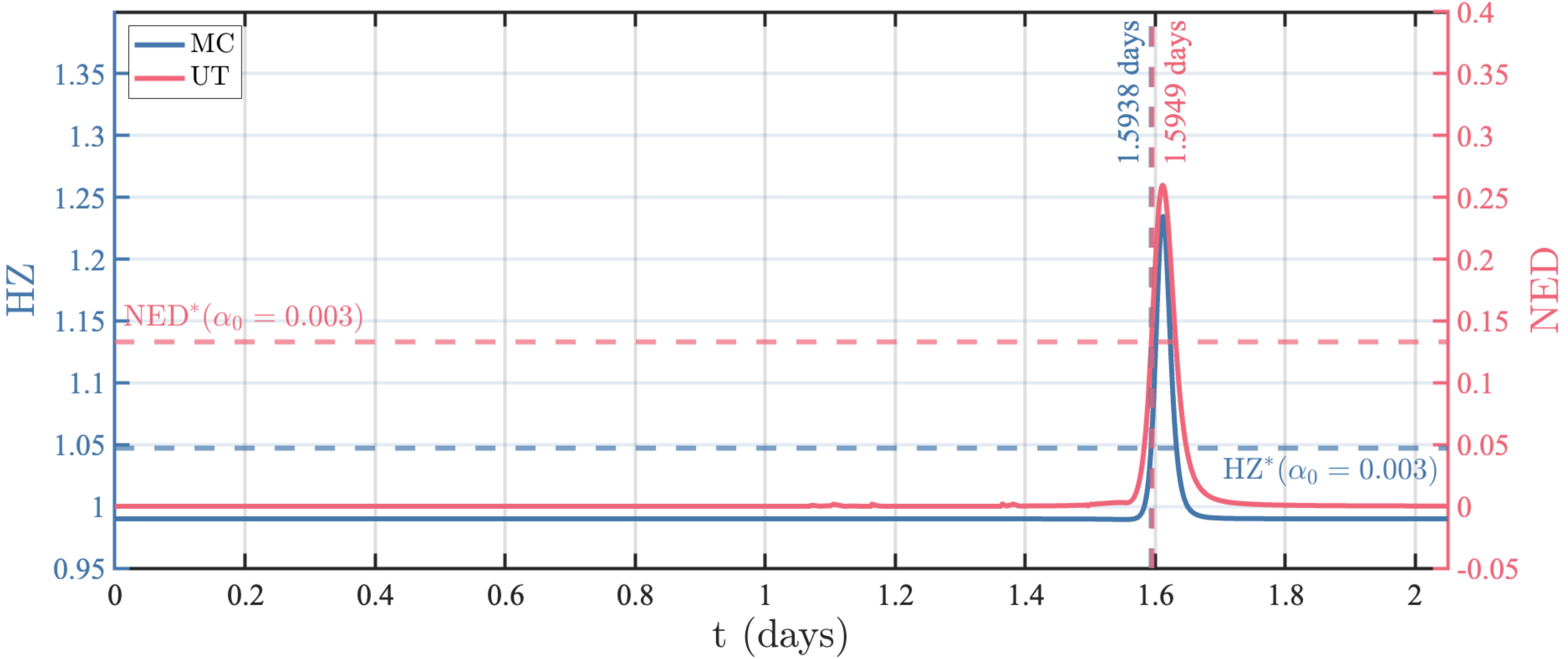
- Second UT-only GVT prediction: **Earth-Moon CAPSTONE 9:2 NRHO**



CAPSTONE trajectory propagated for 8 days, with initial uncertainty $\sigma_r = 100$ m and $\sigma_v = 1$ mm/s.

Utilization of NED GVB For UT-only GVT Prediction

- Second UT-only GVT prediction: **Earth-Moon CAPSTONE 9:2 NRHO**



UT-only GVT prediction is within 1.584 minutes of a MC-based GVT prediction on a completely new trajectory using our derived NED*

Conclusions

Fundamental Questions

1. How do we measure Gaussianity?

$$\mathrm{HZ} = \left[\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\exp\left(-\frac{\beta^2}{2}D_{ij}\right)\right] - \left[2\left(1+\beta^2\right)^{-\frac{d}{2}}\sum_{i=1}^{n}\exp\left(-\frac{\beta^2}{2(1+\beta^2)}D_i\right)\right] + \left[n(1+2\beta^2)^{-\frac{d}{2}}\right]$$

2. How long does it take for state uncertainty to become non-Gaussian?

Family	Distant Prograde	Southern Dragonfly	Northern Dragonfly	Southern Butterfly	Northern Butterfly	L2 Northern Halo	L2 Southern Halo	L1 Southern Halo	L1 Northern Halo	Distant Retrograde
t (periods)	0.022084	0.134874	0.14033	0.14059	0.14536	0.23744	0.23932	0.26968	0.27395	0.51041

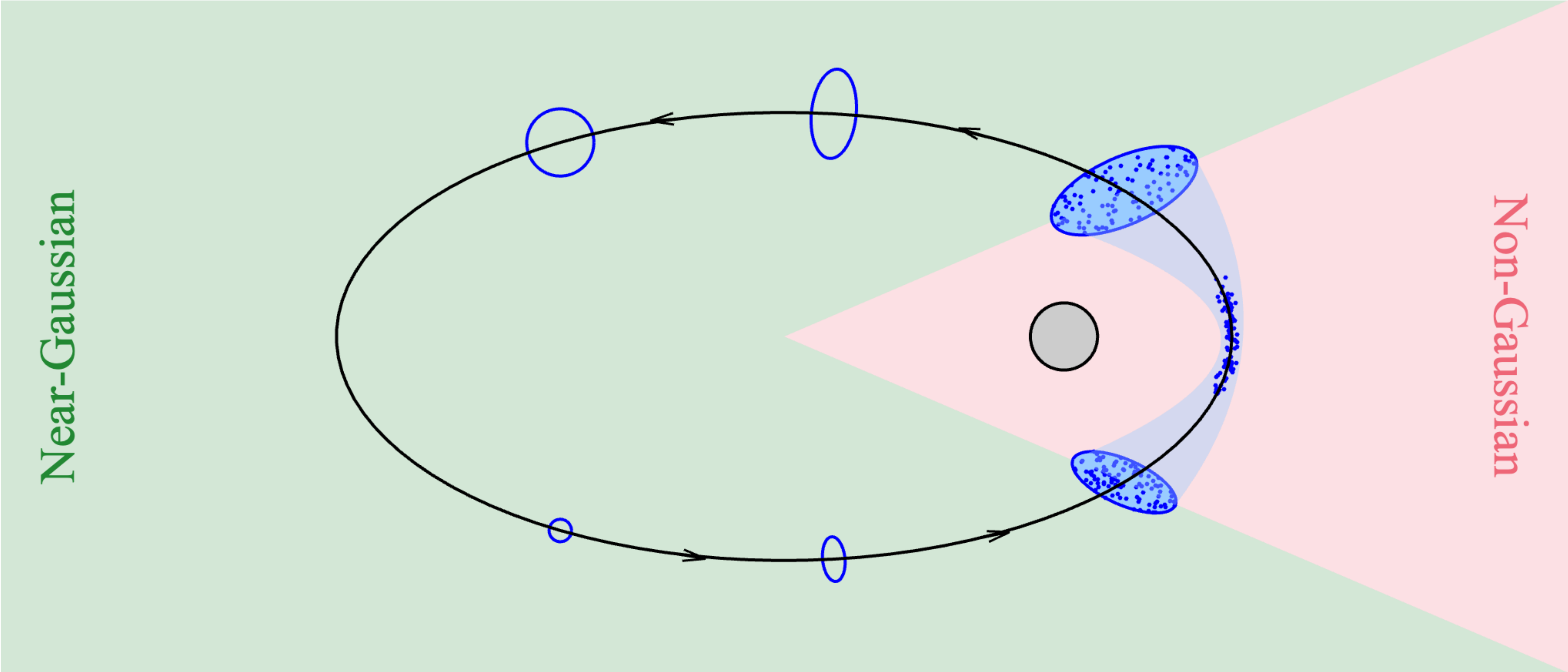
3. Can we predict when state uncertainty is becoming non-Gaussian with an abstraction more efficient to propagate than a dense Monte Carlo?

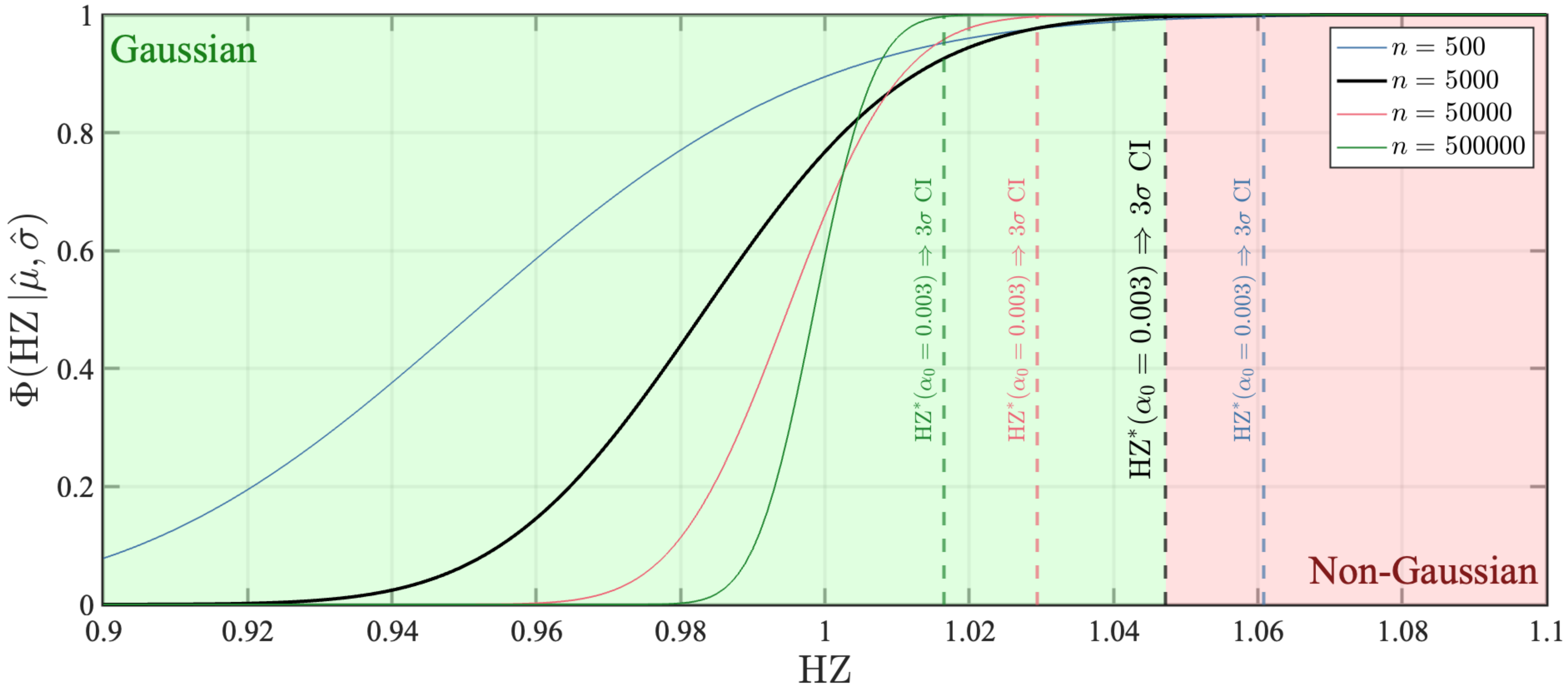
- Using 500 different periodic orbits from the Saturn-Enceladus system, we successfully mapped the NED to the HZ for the CR3BP: $\mathrm{NED}^* = 0.1330 \pm 0.002$
- Performed UT-only GVT predictions of two independent trajectories with **accuracy on the order of minutes**

Future Work

Hybrid Moment/Ensemble Filtering

- Using the NED* value derived in this work, we can develop a hybrid filter that propagates the first and second moments when uncertainty is near-Gaussian, and an ensemble distribution when the uncertainty is non-Gaussian
- Hybrid filter would be more accurate than a pure moment filter and more efficient than a pure ensemble filter





Sparse MC Gaussianity Detection

- NED must be mapped for each uncertainty magnitude and dynamics model, while HZ is a consistent statistic no matter the model or uncertainty
- What are the Type I/II error rates for a sparse MC distribution compared for the large one used in this analysis?

This investigation was supported by the NASA Space Technology Graduate Research Opportunities Fellowship (Grant #80NSSC23K1219)

Thank you for your time. Questions?